Single Image Calibration of Multi-Axial Imaging Systems
Supplementary Materials

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1. Recovering Mirror Parameters with Known Pose and Axis

In this section, we describe in detail the analytical solution for recovering the spherical mirror parameters (radius and distance of center) given checkerboard pose and mirror axis. This is explained in Section 5 in the paper.

When the axis is known, the analysis can be done on the plane of refraction π in 2D for each mirror. Let $[\mathbf{z}_1, \mathbf{z}_2]$ denote an orthogonal coordinate system on π. We choose $\mathbf{z}_2$ along the axis. For a given camera ray $\mathbf{v}(i)$, let $\mathbf{z}_1 = \mathbf{z}_2 \times (\mathbf{z}_2 \times \mathbf{v}(i))$ be the orthogonal direction. The projection of $\mathbf{Q}(i)$ on π is given by $\mathbf{u} = [u_x, u_y]$, where $u_x = \mathbf{z}_1^T \mathbf{Q}(i)$ and $u_y = \mathbf{z}_2^T \mathbf{Q}(i)$. Similarly, the camera ray $\mathbf{v}(i)$ can be represented by the 2D vector $\mathbf{w} = [w_x, w_y]$ on π, where $w_x = \mathbf{z}_1^T \mathbf{v}(i)$ and $w_y = \mathbf{z}_2^T \mathbf{v}(i)$. $\mathbf{w}$ is normalized so that $\mathbf{w}^T\mathbf{w} = 1$. Let $\beta = r^2$.

Referring to Figure 1, the spherical mirror is represented as a circle on π with center at $\mathbf{S} = [0, d]$ and radius r. Let $\mathbf{M} = k\mathbf{w}$ be the common point on the mirror and the camera ray for some scalar k. Since $\mathbf{M}$ lies on the circle, $(\mathbf{M} - \mathbf{S})^T(\mathbf{M} - \mathbf{S}) = r^2 = \beta$. Hence,

$$(kw_x)^2 + (kw_y - d)^2 = k^2 - 2kw_y d + d^2 = \beta. \tag{1}$$

The normal at $\mathbf{M}$ is given by $\mathbf{n} = \mathbf{M} - \mathbf{S}$, where $\mathbf{n}^T\mathbf{n} = \beta$. The reflected ray $\mathbf{v}_1$ is given by the Snell's law as

$$\mathbf{v}_1 = \mathbf{w} - 2\mathbf{n}(\mathbf{n}^T\mathbf{w})/\beta. \tag{2}$$

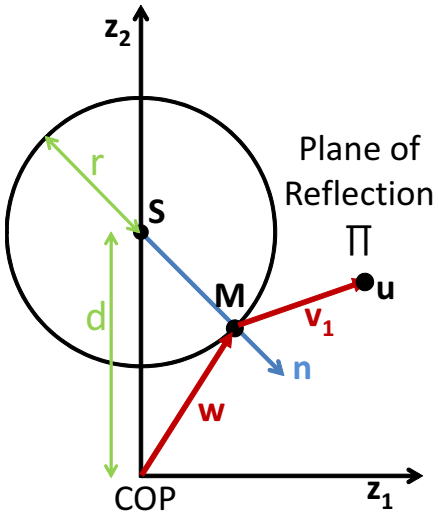

Figure 1. Estimation of sphere radius r and distance d for spherical mirror.

The reflected ray should pass through $\mathbf{u}$. Therefore,

$$0 = \mathbf{v}_1 \times (\mathbf{u} - \mathbf{M}). \tag{3}$$

After substituting all the terms, the above equation simplifies to

$$K_1 k^2 + K_2 k + K_3 = 0, \tag{4}$$

where

$$\begin{aligned} K_1 &= 2(dw_x + u_x w_y - u_y w_x) && (5)\\ K_2 &= -2d(u_x + u_x w_y^2 + dw_x w_y - u_y w_x w_y) && (6)\\ K_3 &= 2u_x w_y d^2 - \beta u_x w_y + \beta u_y w_x && (7) \end{aligned}$$

Now we eliminate k between (1) and (4). Removing k^2 term between (1) and (4), we can solve for k as

$$k = \frac{(-2w_x d^3 + 2u_y w_x d^2 + 2\beta w_x d + \beta u_x w_y - \beta u_y w_x)}{(-2w_x d^2 w_y - 2u_x d w_y^2 + 2u_y w_x d w_y + 2u_x d)} \tag{8}$$

Substituting k back into (1), we get the following equation in d and β

$$EQ_1: \quad 0 = c_0 + c_1 d + c_2 d^2 + c_3 d^3 + c_4 d^4 + c_5 d^5 + c_6 d^6, \tag{9}$$

where,

$$\begin{aligned} c_6 &= -4w_x^2(w_y - 1)(w_y + 1)\\ c_5 &= 8u_y w_x^2(w_y - 1)(w_y + 1)\\ c_4 &= 4(u_x^2 w_y^4 - 2u_x^2 w_y^2 + u_x^2 - u_y^2 w_x^2 w_y^2 + u_y^2 w_x^2 + \beta w_x^2 w_y^2 - 2\beta w_x^2)\\ c_3 &= 4\beta w_x(u_x w_y^3 - u_y w_x w_y^2 - u_x w_y + 3u_y w_x)\\ c_2 &= 4\beta(u_x^2 w_y^2 - u_x^2 - u_y^2 w_x^2 + \beta w_x^2)\\ c_1 &= 4\beta^2 w_x(u_x w_y - u_y w_x)\\ c_0 &= \beta^2(u_x w_y - u_y w_x)^2 \end{aligned}$$

This equation is 6^{th} degree in d and quadratic in β. Since we have two unknowns (d and β), we need two equations from two correspondences. Let EQ_1 and EQ_2 be the two such equations. By eliminating β between EQ_1 and EQ_2, we obtain a 7^{th} degree equation.

Re-write the above equation in powers of β, we get

$$\begin{aligned} EQ_1: &\quad 0 = K_{11} + K_{12}\beta + K_{13}\beta^2 && (10)\\ EQ_2: &\quad 0 = K_{21} + K_{22}\beta + K_{23}\beta^2 && (11)\\ & && (12) \end{aligned}$$

where K_{ij} are constants. Eliminating β^2, we get

$$\beta = \frac{K_{21}K_{13} - K_{11}K_{23}}{K_{12}K_{23} - K_{22}K_{13}}. \tag{13}$$

$$\tag{14}$$

Substituting β back into EQ_1 or EQ_2 and simplifying gives the 7^{th} degree equation in single unknown d.

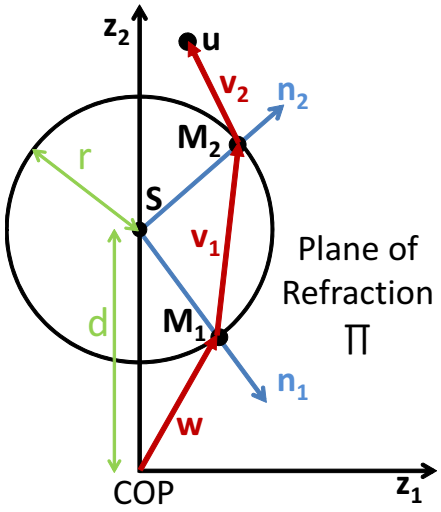

Figure 2. Estimation of sphere radius r and distance d for refractive sphere.

2. Details on Recovering Refractive Sphere Parameters

Now we consider a camera looking through multiple refractive spheres (ball lenses). Similar to mirrors, the axis is defined as the vector joining the camera center to the center of the refractive sphere. The calibration parameters consist of centers, radii and refractive indices of the balls along with the unknown checkerboard pose. We assume that the refractive index μ is known and is identical for all spheres. As described in the paper, we can compute the checkerboard pose and axis of each refractive ball in exactly the same manner using the multi-axial property. Only the estimation of ball center and radius is different from the previous case due to refraction.

As in before, the analysis can be done on the plane of refraction π (see Figure 2), since the entire light path remains on π. Let $\mathbf{M}_1 = k\mathbf{w}$ be the common point on the refractive ball and the camera ray $\mathbf{w}$. Then we have

$$(kw_x)^2 + (kw_y - d)^2 = k^2 - 2kw_y d + d^2 = \beta, \tag{15}$$

similar to (1). The normal at $\mathbf{M}_1$ is given by $\mathbf{n}_1 = \mathbf{M}_1 - \mathbf{S}$, where $\mathbf{n}_1^T\mathbf{n}_1 = \beta$. The first refracted ray $\mathbf{v}_1$ is given by the Snell's law [1] as

$$\mathbf{v}_1 = \frac{1}{\mu}\mathbf{w} + b_1\mathbf{n}_1, \tag{16}$$

where

$$b_1 = \frac{-\mathbf{w}^T\mathbf{n}_1 - \sqrt{(\mathbf{w}^T\mathbf{n}_1)^2 - (1-\mu^2)\beta}}{\mu\beta}. \tag{17}$$

Now $\mathbf{w}^T\mathbf{n}_1 = k\mathbf{w}^T\mathbf{w} - \mathbf{w}^T\mathbf{S} = k - dw_y$. Inserting terms, we get

$$b_1 = -\frac{k - dw_y + \sqrt{(k-dw_y)^2 + \beta(\mu^2-1)}}{\beta\mu} \tag{18}$$

The second refraction point $\mathbf{M}_2$ can be written as $\mathbf{M}_2 = \mathbf{M}_1 + \lambda\mathbf{v}_1$ for some constant λ. Since $\mathbf{M}_2$ lies on the sphere,

$$\begin{aligned} \beta &= (\mathbf{M}_2 - \mathbf{S})^T(\mathbf{M}_2 - \mathbf{S}) = (\mathbf{n}_1 + \lambda\mathbf{v}_1)^T(\mathbf{n}_1 + \lambda\mathbf{v}_1), \\ &= \beta + 2\lambda\mathbf{n}_1^T\mathbf{v}_1 + \lambda^2\mathbf{v}_1^T\mathbf{v}_1. \end{aligned} \tag{19}$$

This gives

$$\lambda = -2\mathbf{n}_1^T\mathbf{v}_1/(\mathbf{v}_1^T\mathbf{v}_1) \tag{20}$$

$$= \frac{2\sqrt{d^2w_y^2 - 2dkw_y + k^2 + \beta\mu^2 - \beta}}{\mu} \tag{21}$$

The outgoing ray $\mathbf{v}_2$ can be written as

$$\mathbf{v}_2 = \mu\mathbf{v}_1 + b_2\mathbf{n}_2 \tag{22}$$

for some b_2. Taking dot product with $\mathbf{n}_2$ on both sides,

$$b_2 = (\mathbf{v}_2^T\mathbf{n}_2 - \mu\mathbf{v}_1^T\mathbf{n}_2)/\beta \tag{23}$$

From the symmetry of the sphere, we have

$$\mathbf{v}_2^T\mathbf{n}_2 = -\mathbf{w}^T\mathbf{n}_1. \tag{24}$$

and

$$\mathbf{v}_1^T\mathbf{n}_2 = -\mathbf{v}_1^T\mathbf{n}_1. \tag{25}$$

Thus,

$$b_2 = (-\mathbf{w}^T\mathbf{n}_1 + \mu\mathbf{v}_1^T\mathbf{n}_1)/\beta \tag{26}$$

$$= -\frac{k - dw_y + \sqrt{d^2w_y^2 - 2dkw_y + k^2 + \beta\mu^2 - \beta}}{\mu} \tag{27}$$

Finally, the outgoing ray $\mathbf{v}_2$ should pass through $\mathbf{u}$. Thus,

$$\mathbf{v}_2 \times (\mathbf{u} - \mathbf{M}_2) = 0. \tag{28}$$

After substituting all the terms, the above equation simplifies to following equation which is 5^{th} degree in d, 4^{th} degree in k and quadratic in β. The equation can be written as

$$c_0 + c_1k + c_2k^2 + c_3k^3 + c_4k^4 = 0 \tag{29}$$

where

$$
\begin{aligned}
c_4 &= 4(dw_x + u_x w_y - u_y w_x)(dw_x + u_x w_y - u_y w_x - 2d\mu^2 w_x + d\mu^4 w_x + \mu^4 u_x w_y - \mu^4 u_y w_x) \\
c_3 &= -8d(d^2\mu^4 w_x^2 w_y - 3d^2\mu^2 w_x^2 w_y + 2d^2 w_x^2 w_y + 2d\mu^4 u_x w_x w_y^2 + d\mu^4 u_x w_x - 2d\mu^4 u_y w_x^2 w_y - 3d\mu^2 u_x w_x w_y^2 \\
&\quad -d\mu^2 u_x w_x + 3d\mu^2 u_y w_x^2 w_y + 4du_x w_x w_y^2 - 4du_y w_x^2 w_y + \mu^4 u_x^2 w_y^3 + \mu^4 u_x^2 w_y - 2\mu^4 u_x u_y w_x w_y^2 \\
&\quad -\mu^4 u_x u_y w_x + \mu^4 u_y^2 w_x^2 w_y + 2u_x^2 w_y^3 - 4u_x u_y w_x w_y^2 + 2u_y^2 w_x^2 w_y) \\
c_2 &= -4(-d^4\mu^4 w_x^2 w_y^2 + 6d^4\mu^2 w_x^2 w_y^2 - 5d^4 w_x^2 w_y^2 - 2d^3\mu^4 u_x w_x w_y^3 - 4d^3\mu^4 u_x w_x w_y + 2d^3\mu^4 u_y w_x^2 w_y^2 + 6d^3\mu^2 u_x w_x w_y^3 \\
&\quad +6d^3\mu^2 u_x w_x w_y - 6d^3\mu^2 u_y w_x^2 w_y^2 - 10d^3 u_x w_x w_y^3 - 2d^3 u_x w_x w_y + 10d^3 u_y w_x^2 w_y^2 - d^2\mu^4 u_x^2 w_y^4 - 4d^2\mu^4 u_x^2 w_y^2 \\
&\quad -d^2\mu^4 u_x^2 + 2d^2\mu^4 u_x u_y w_x w_y^3 + 4d^2\mu^4 u_x u_y w_x w_y - d^2\mu^4 u_y^2 w_x^2 w_y^2 + 2\beta d^2\mu^4 w_x^2 - 4\beta d^2\mu^2 w_x^2 \\
&\quad -5d^2 u_x^2 w_y^4 - 2d^2 u_x^2 w_y^2 + d^2 u_x^2 + 10d^2 u_x u_y w_x w_y^3 + 2d^2 u_x u_y w_x w_y - 5d^2 u_y^2 w_x^2 w_y^2 + 2\beta d^2 w_x^2 \\
&\quad +3\beta d\mu^4 u_x w_x w_y - 3\beta d\mu^4 u_y w_x^2 - 5\beta d\mu^2 u_x w_x w_y + 5\beta d\mu^2 u_y w_x^2 + 4\beta du_x w_x w_y - 4\beta du_y w_x^2 + \beta\mu^4 u_x^2 w_y^2 \\
&\quad -2\beta\mu^4 u_x u_y w_x w_y + \beta\mu^4 u_y^2 w_x^2 - \beta\mu^2 u_x^2 w_y^2 + 2\beta\mu^2 u_x u_y w_x w_y - \beta\mu^2 u_y^2 w_x^2 + 2\beta u_x^2 w_y^2 \\
&\quad -4\beta u_x u_y w_x w_y + 2\beta u_y^2 w_x^2) \\
c_1 &= 4d(2d^4\mu^2 w_x^2 w_y^3 - 2d^4 w_x^2 w_y^3 - 2d^3\mu^4 u_x w_x w_y^2 + 2d^3\mu^2 u_x w_x w_y^4 + 6d^3\mu^2 u_x w_x w_y^2 - 2d^3\mu^2 u_y w_x^2 w_y^3 \\
&\quad -4d^3 u_x w_x w_y^4 - 4d^3 u_x w_x w_y^2 + 4d^3 u_y w_x^2 w_y^3 - 2d^2\mu^4 u_x^2 w_y^3 - 2d^2\mu^4 u_x^2 w_y + 2d^2\mu^4 u_x u_y w_x w_y^2 + 2\beta d^2\mu^4 w_x^2 w_y \\
&\quad -6\beta d^2\mu^2 w_x^2 w_y - 2d^2 u_x^2 w_y^5 - 4d^2 u_x^2 w_y^3 + 2d^2 u_x^2 w_y + 4d^2 u_x u_y w_x w_y^4 + 4d^2 u_x u_y w_x w_y^2 - 2d^2 u_y^2 w_x^2 w_y^3 + 4\beta d^2 w_x^2 w_y \\
&\quad +3\beta d\mu^4 u_x w_x w_y^2 + 2\beta d\mu^4 u_x w_x - 3\beta d\mu^4 u_y w_x^2 w_y - 8\beta d\mu^2 u_x w_x w_y^2 - 2\beta d\mu^2 u_x w_x + 8\beta d\mu^2 u_y w_x^2 w_y \\
&\quad +8\beta du_x w_x w_y^2 - 8\beta du_y w_x^2 w_y \\
&\quad +\beta\mu^4 u_x^2 w_y^3 + \beta\mu^4 u_x^2 w_y - 2\beta\mu^4 u_x u_y w_x w_y^2 - \beta\mu^4 u_x u_y w_x + \beta\mu^4 u_y^2 w_x^2 w_y - 2\beta\mu^2 u_x^2 w_y^3 \\
&\quad +4\beta\mu^2 u_x u_y w_x w_y^2 - 2\beta\mu^2 u_y^2 w_x^2 w_y + 4\beta u_x^2 w_y^3 - 8\beta u_x u_y w_x w_y^2 + 4\beta u_y^2 w_x^2 w_y) \\
c_0 &= 4\beta^2 d^2\mu^4 w_x^2 - 8\beta^2 d^2\mu^2 w_x^2 + 4\beta^2 d^2 w_x^2 + 4\beta^2 d\mu^4 u_x w_x w_y - 4\beta^2 d\mu^4 u_y w_x^2 \\
&\quad -12\beta^2 d\mu^2 u_x w_x w_y + 12\beta^2 d\mu^2 u_y w_x^2 + 8\beta^2 du_x w_x w_y - 8\beta^2 du_y w_x^2 + \beta^2\mu^4 u_x^2 w_y^2 \\
&\quad -2\beta^2\mu^4 u_x u_y w_x w_y + \beta^2\mu^4 u_y^2 w_x^2 - 4\beta^2\mu^2 u_x^2 w_y^2 + 8\beta^2\mu^2 u_x u_y w_x w_y \\
&\quad -4\beta^2\mu^2 u_y^2 w_x^2 + 4\beta^2 u_x^2 w_y^2 - 8\beta^2 u_x u_y w_x w_y + 4\beta^2 u_y^2 w_x^2 + 4\beta d^4\mu^2 w_x^2 w_y^2 \\
&\quad -4\beta d^4 w_x^2 w_y^2 - 8\beta d^3\mu^4 u_x w_x w_y + 4\beta d^3\mu^2 u_x w_x w_y^3 + 16\beta d^3\mu^2 u_x w_x w_y - 4\beta d^3\mu^2 u_y w_x^2 w_y^2 \\
&\quad -8\beta d^3 u_x w_x w_y^3 - 8\beta d^3 u_x w_x w_y + 8\beta d^3 u_y w_x^2 w_y^2 - 4\beta d^2\mu^4 u_x^2 w_y^2 + 4\beta d^2\mu^4 u_x u_y w_x w_y \\
&\quad +8\beta d^2\mu^2 u_x^2 w_y^2 - 4\beta d^2\mu^2 u_x^2 - 8\beta d^2\mu^2 u_x u_y w_x w_y - 4\beta d^2 u_x^2 w_y^4 - 8\beta d^2 u_x^2 w_y^2 + 4\beta d^2 u_x^2 \\
&\quad +8\beta d^2 u_x u_y w_x w_y^3 + 8\beta d^2 u_x u_y w_x w_y - 4\beta d^2 u_y^2 w_x^2 w_y^2 - 8d^5\mu^2 u_x w_x w_y^3 + 8d^5 u_x w_x w_y^3 + 4d^4\mu^4 u_x^2 w_y^2 \\
&\quad +8d^4 u_x^2 w_y^4 - 4d^4 u_x^2 w_y^2 - 8d^4 u_x u_y w_x w_y^3
\end{aligned}
$$

Eliminating k between this equation and (15), we obtain a single equation in d and β, which is 12^{th} degree in d and fourth degree in β.

References

[1] A. Agrawal, Y. Taguchi, and S. Ramalingam. Analytical forward projection for axial non-central dioptric and catadioptric cameras. In *ECCV*, volume 6313, pages 129–143, Sept. 2010.