Multi-View Geometry and Computational Photography Using Non-Classical Cameras

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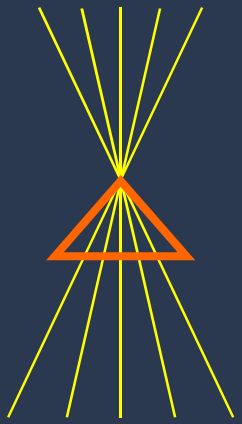
Schedule

Introduction	10 mins
Geometry of Rays & Applications	40 mins
Break	10 min
Modeling	60 mins
Break	10 min
Synthesis/Reconstructions	45 mins
Future Directions	15 mins
Discussion	

Modeling of Cameras

- Forward Projection
  - Computing the pixel corresponding to a 3D point
  - Analytical solutions, optimization, GLC
- Back Projection
  - Computing the 3D ray corresponding to a pixel
  - Same as Calibration

Perspective Cameras (Central)



Single Viewpoint
(Central)

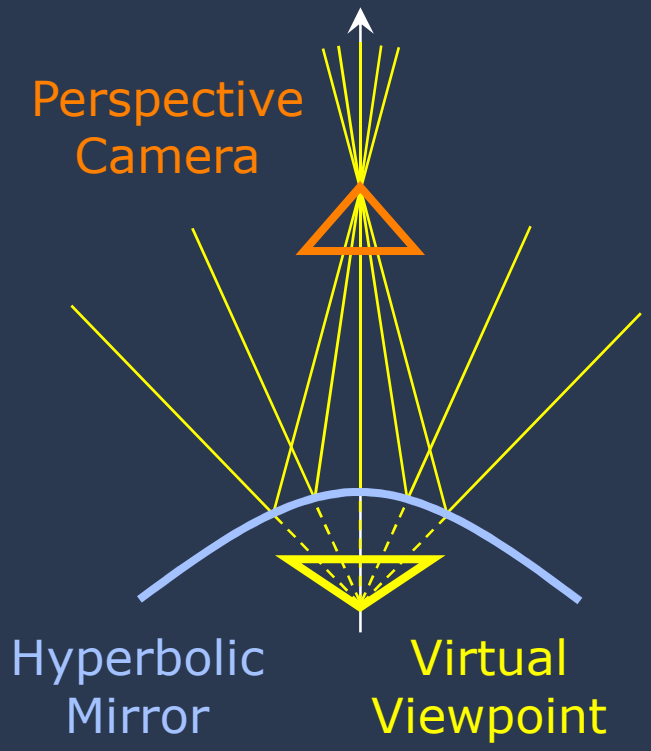
Perspective Camera

Single Viewpoint

Single-Viewpoint Catadioptric Cameras

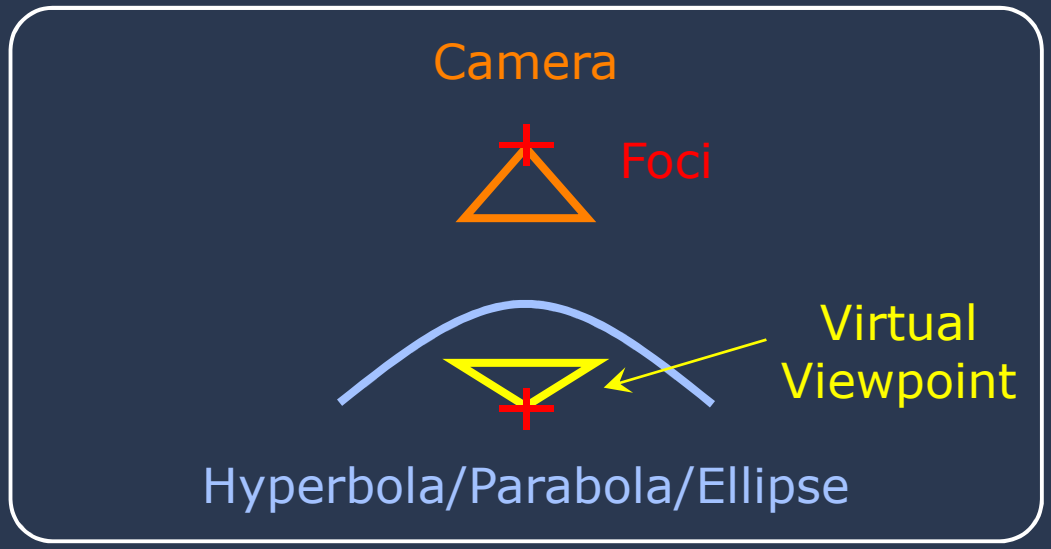
- Mirror + Perspective Camera
- Wide Field of View

Single Viewpoint
(Central)



Single-Viewpoint Catadioptric Cameras

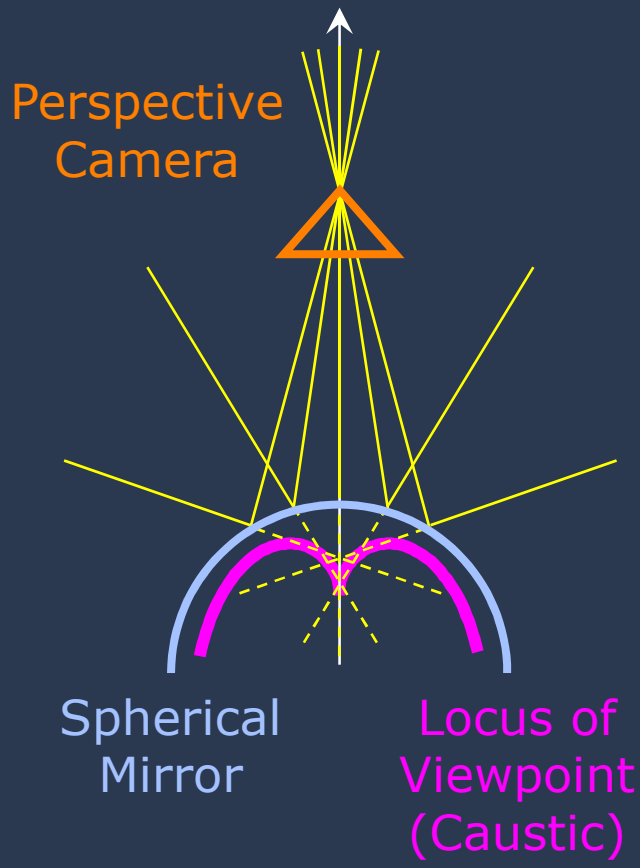
[Baker & Nayar 99]



- Only a few single-viewpoint configurations
- Other configurations lead to non-single viewpoint
  - Spherical mirror
  - Camera not on foci
  - Multiple mirrors

Non-Central Catadioptric Cameras

Camera looking into four spherical mirrors

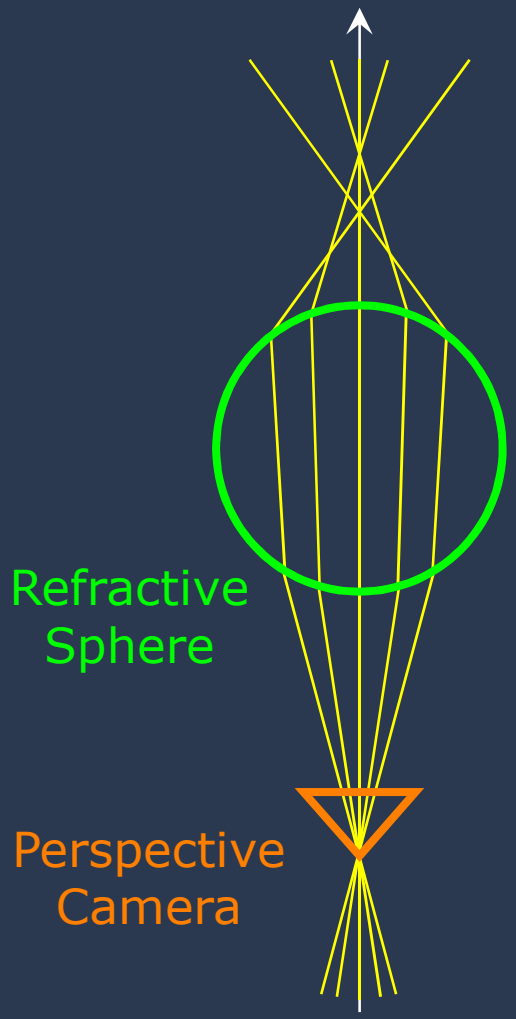


Can we analytically model the projection of 3D points to pixels?

Convention Center
creta maris

Non-Central Dioptric Camera

- Looking through a refractive glass sphere
- Google “Crystal Ball Photography”

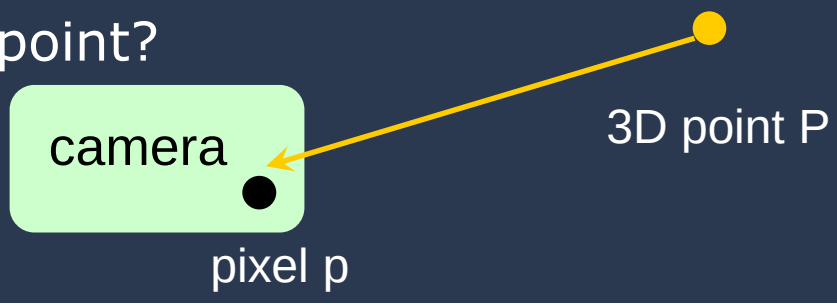


Why are non-central cameras difficult to model?

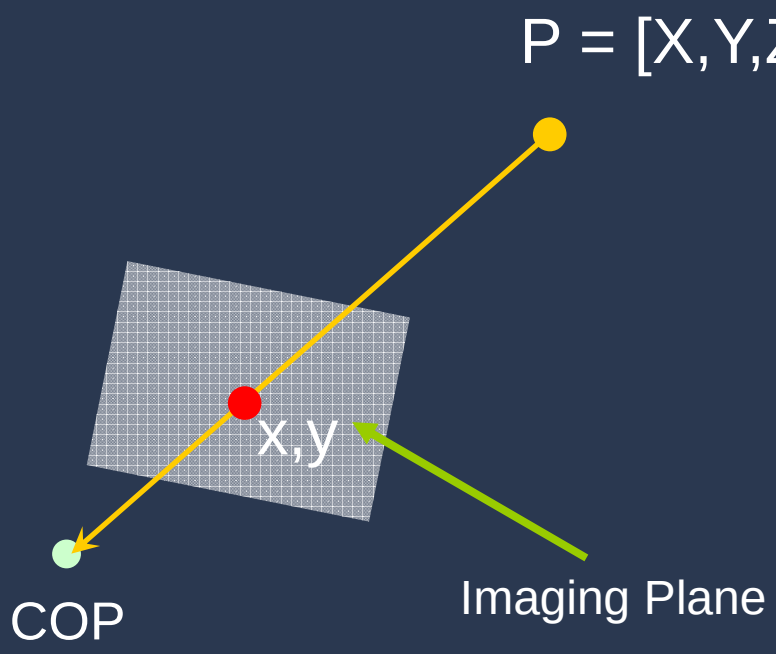
- Following two operations are essential for any camera

- Back Projection
  - What is the 3D ray corresponding to a pixel?
  - Generic Camera Calibration
    - Grossberg and Nayar, ICCV 2001
    - Sturm & Ramalingam, ECCV 2004
    - Ramalingam et al. CVPR 2005

- Forward Projection
  - What is the projection of a 3D point?
  - Inverse Ray Tracing
  - Compute the Light-Path



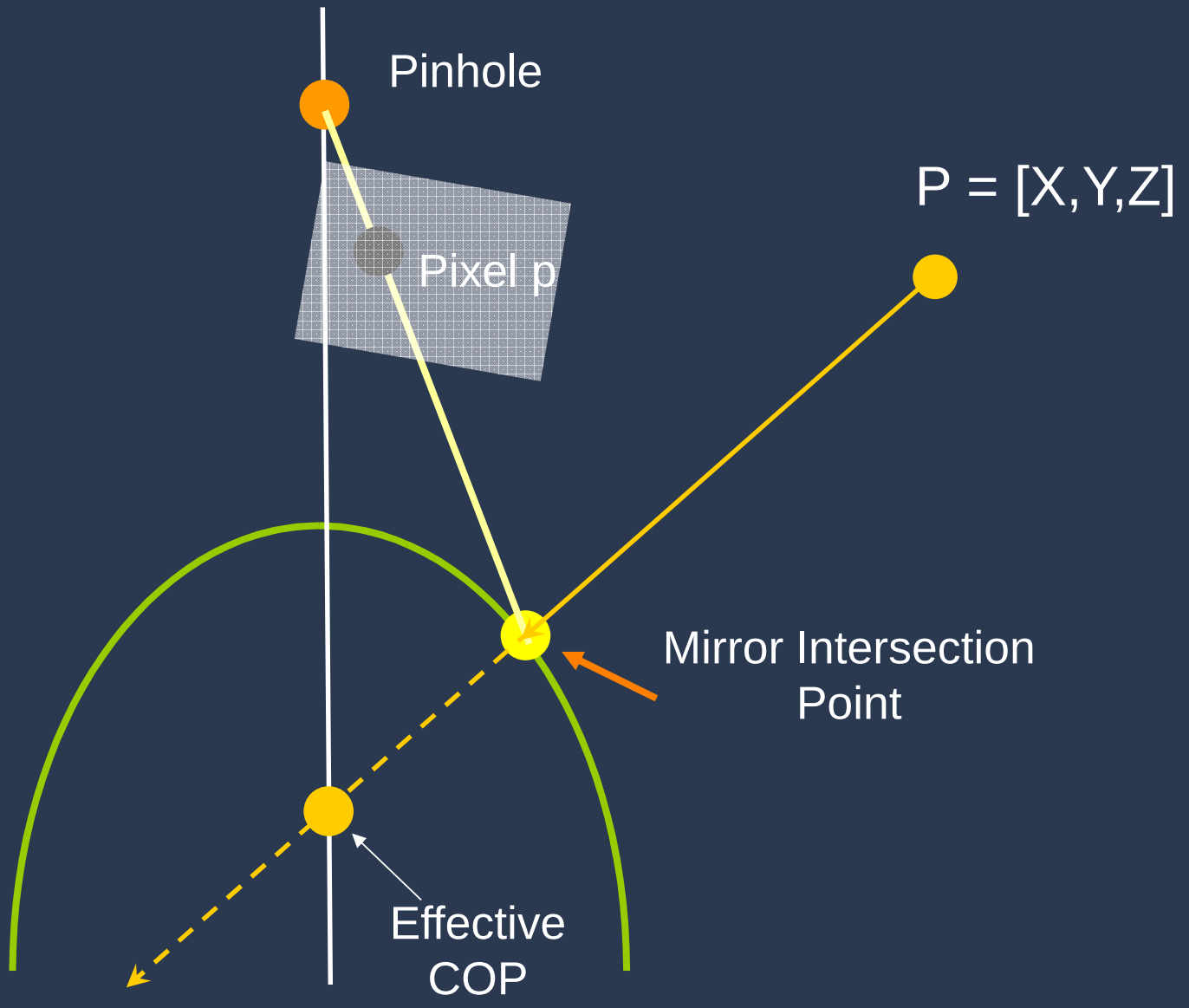
Forward Projection: Easy for Perspective Camera



$x = f*X/Z$

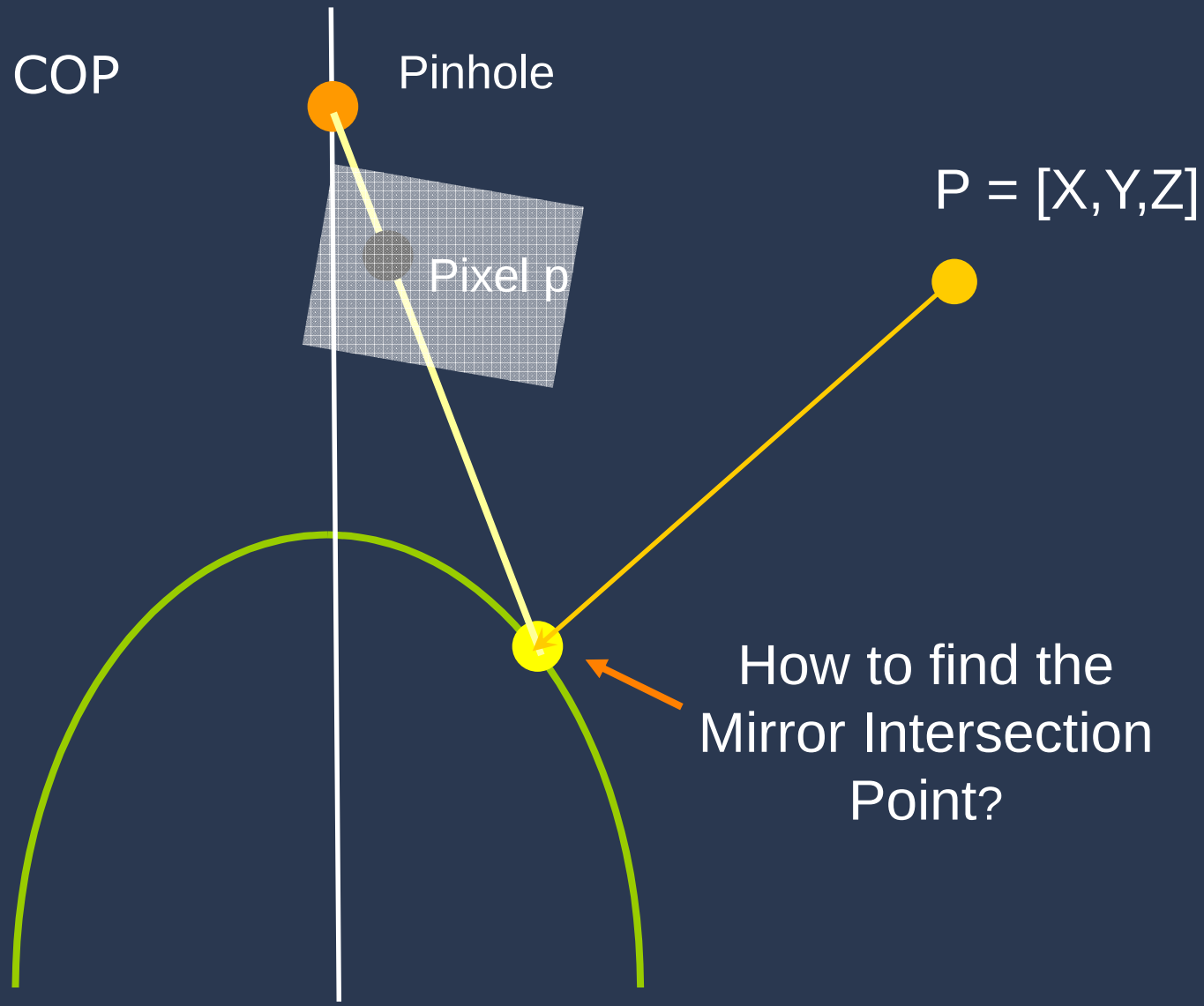
$y = f*Y/Z$

Easy for Central Catadioptric Camera



But, difficult for Non-Central Camera

- No effective COP



Forward Projection for Non-Central Camera

- Given a 3D point P
  - What is the corresponding image pixel p?
- Difficult, no closed form solution
- Can do optimization (Micusik and Pajdla, CVPR 2004)
  - Iterative Forward Projection
  - Slow

Can we obtain analytical solution for forward projection for non-central cameras?

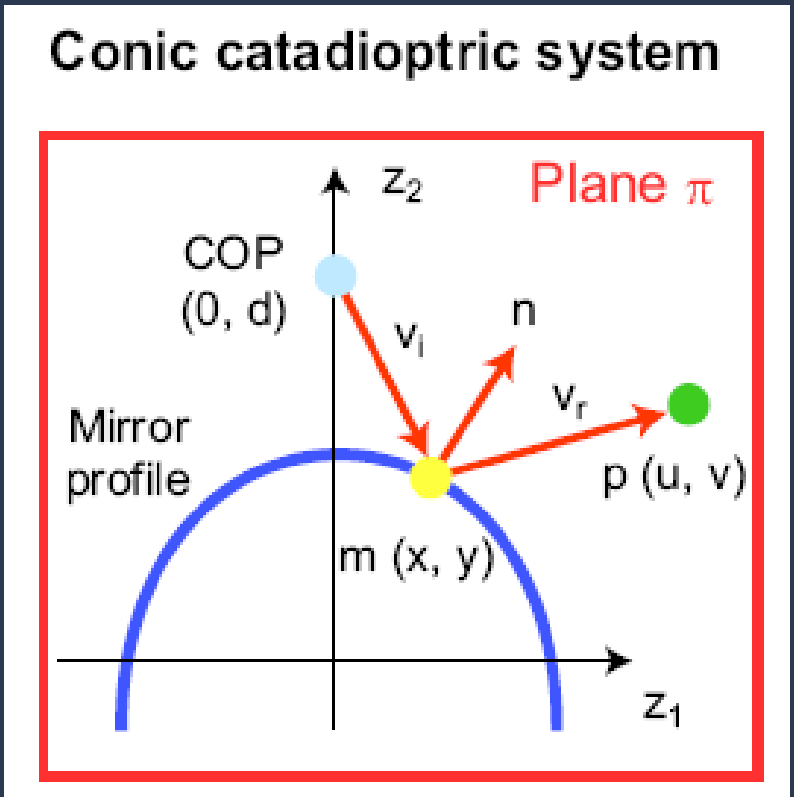
Why Forward Projection?

- Allows using *existing* bundle-adjustment algorithms for non-classical cameras

- Simply replace the perspective projection with analytical forward projection!
  - Image projection of 3D point allows minimizing re-projection error

Iterative Forward Projection

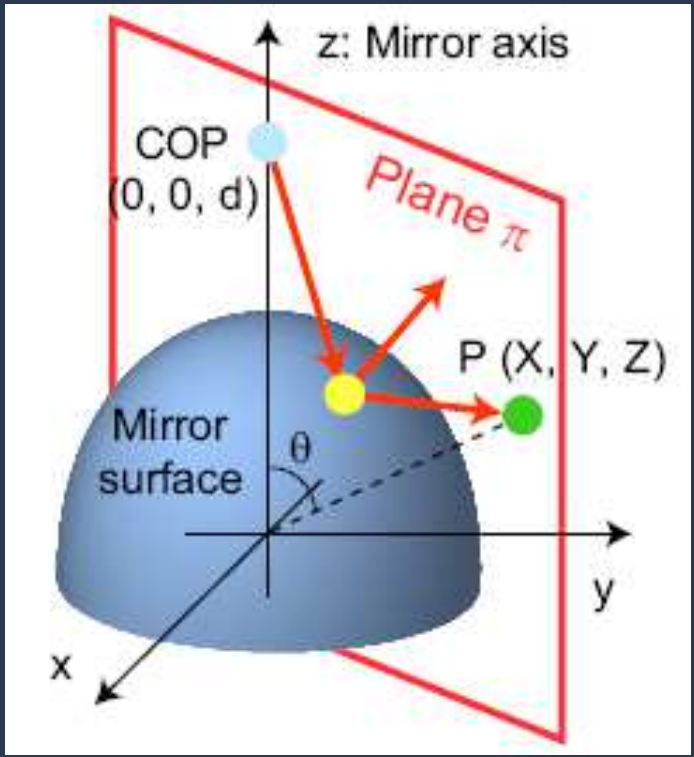
- Assume central model
- Find an initial solution
- Optimize by minimizing the cost based on Snell's law

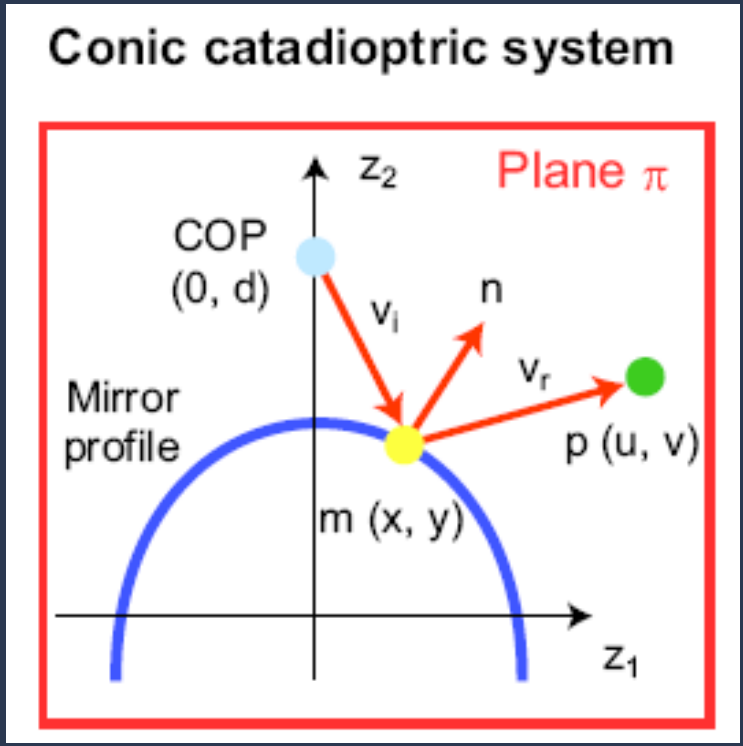


Analytical Forward Projection for Non-Central Cameras

- Axial Configuration
  - Camera lies on the axis of rotationally symmetric mirror
- For conic mirrors
  - Solve 6^{th} degree equation in one unknown
  - Reduces to 4^{th} degree equation for spherical mirror
    - Closed Form Solution
- For refractive sphere
  - Solve 10^{th} degree equation in one unknown
- 100 times speed up
  - 3D reconstruction using bundle adjustment

Finding the Mirror Intersection Point





$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

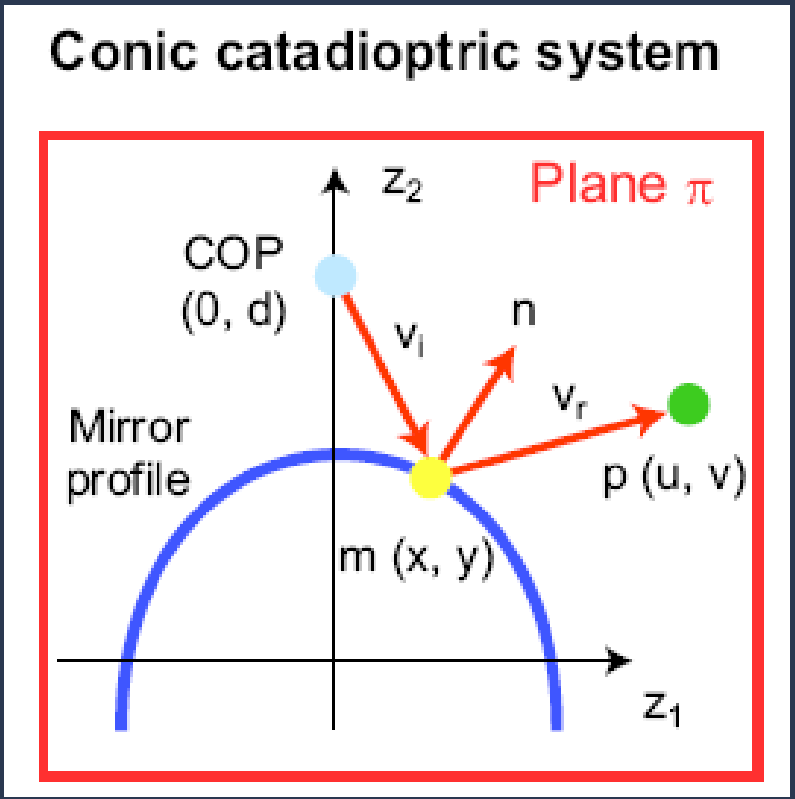
Constraints

- [x,y] lies on the mirror

$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

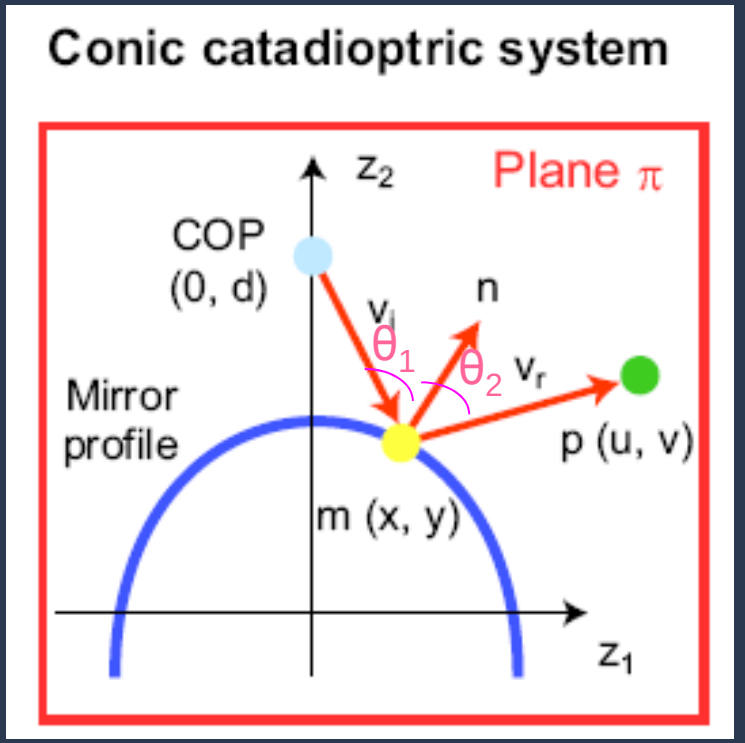
$$x = \pm\sqrt{C - By - Ay^2}$$



Constraints

- Planarity
  - Incoming ray (v_i), normal (n) and reflected ray (v_r) lie on same plane
- Angle constraint
  - $\theta_1 = \theta_2$
- Use vector form of law of reflection

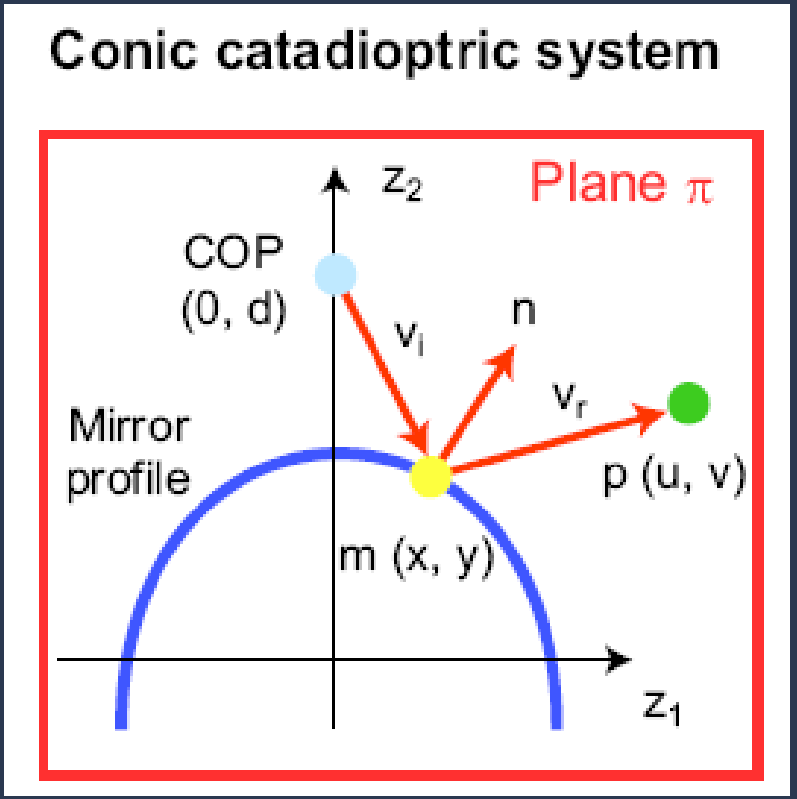
$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{n}^T\mathbf{v}_i)/(\mathbf{n}^T\mathbf{n}).$$



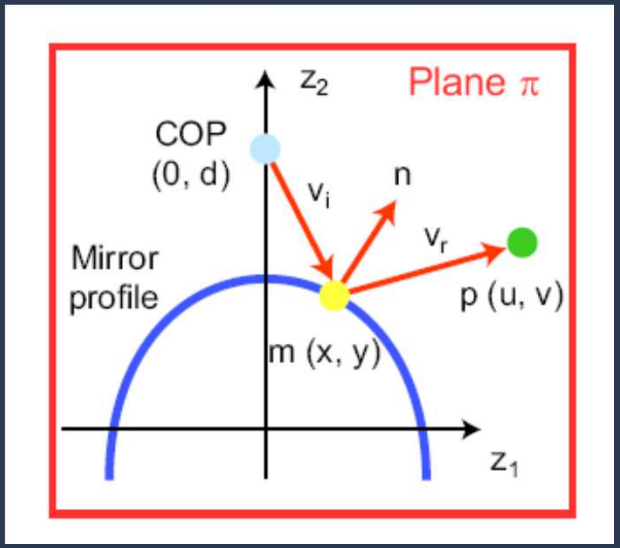
Constraint

- The reflected ray should pass through the given point

- Cross(v_r , p – m) = 0



Solution



$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

$$x = \pm\sqrt{C - By - Ay^2}$$

$$\mathbf{n} = \begin{bmatrix} x \\ B/2 + Ay \end{bmatrix}, \quad \mathbf{v}_i = \begin{bmatrix} x \\ y - d \end{bmatrix}$$

$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{n}^T\mathbf{v}_i)/(\mathbf{n}^T\mathbf{n}).$$

$$\mathbf{v}_r \times (\mathbf{p} - \mathbf{m}) = 0$$

Solution

$$u^2 K_1^2(y) + K_2^2(y)(Ay^2 + By - C) = 0,$$

$$K_1(y) = K_{11}y^3 + K_{12}y^2 + K_{13}y + K_{14}$$

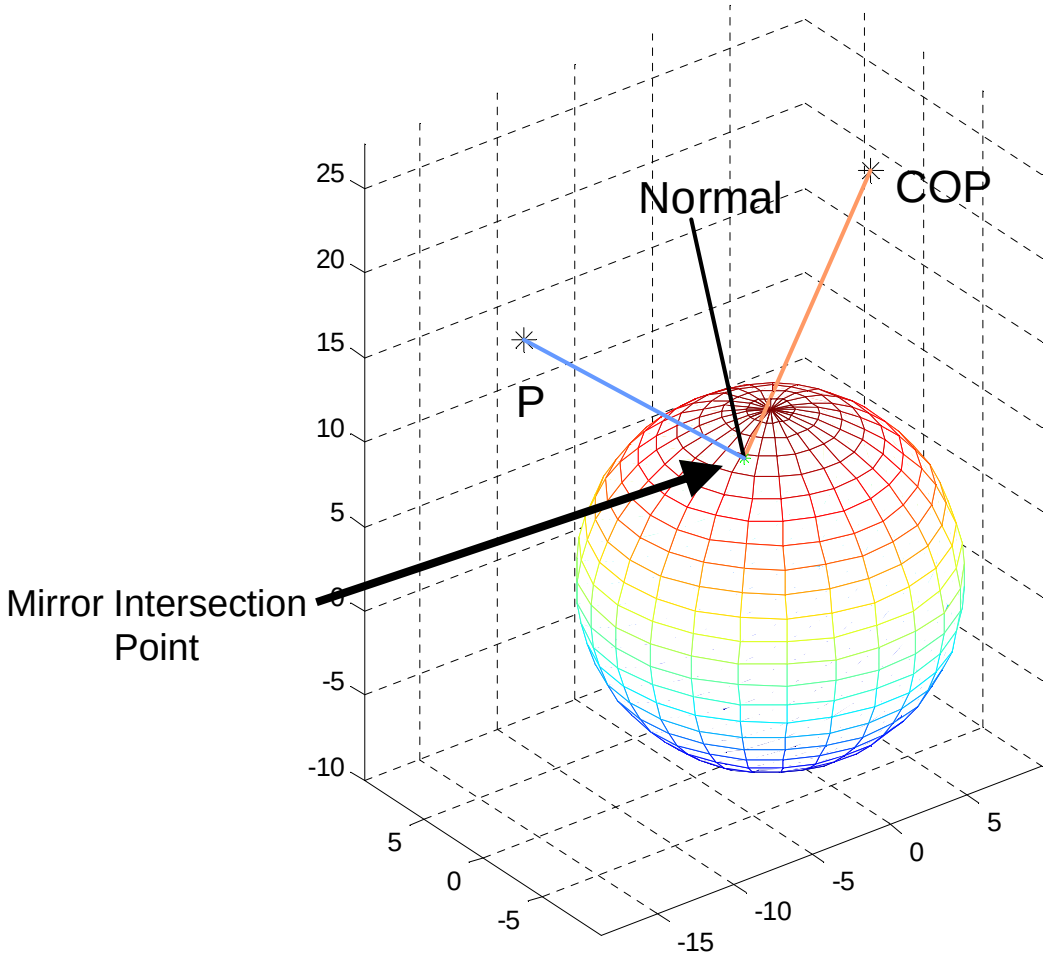
$$K_2(y) = K_{21}y^2 + K_{22}y + K_{23}$$

- 6th degree equation in y
  - 6 solutions
  - Get the correct solution by checking law of reflection
- Obtain x using $x = \pm\sqrt{C - By - Ay^2}$
- 20 lines of Matlab Code

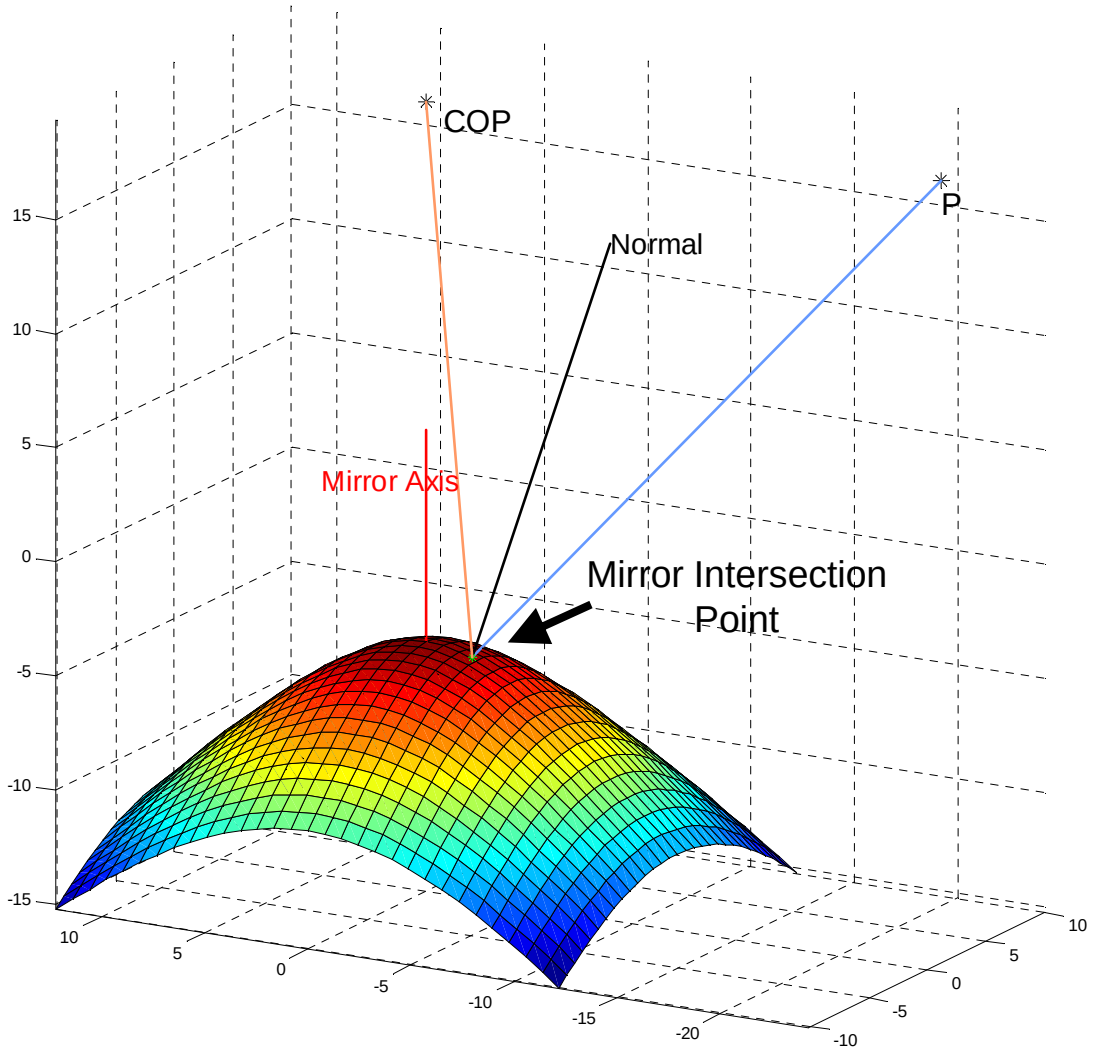
Mirror Shape $Az_1^2+z_2^2+Bz_1=C$

Mirror Shape	Pinhole Placement	Parameters	Central	System Degree
General	On axis	A,B,C	No	6
Sphere	Any	$A=1, B=0, C>0$	No	4
Elliptic	On axis, At Foci	$B=0$	Yes	2
Elliptic	On axis, Not at Foci	$B=0$	No	6
Hyperbolic	On axis, At Foci	$A<0, C<0$	Yes	2
Hyperbolic	On axis, Not at Foci	$A<0, C<0$	No	6
Parabolic	On axis, $d=\infty$	$A=0, C=0$	Yes	2
Parabolic	On axis, Finite d	$A=0, C=0$	No	5

Degree of Forward Projection Equation for Various Configurations



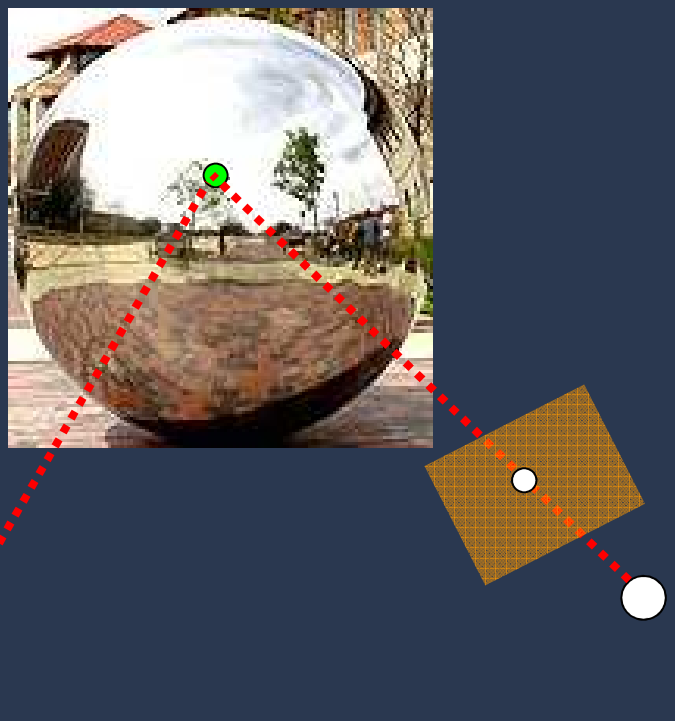
Visualization for Spherical Mirror



Visualization for Hyperbolic Mirror

150 A.D., Ptolemy

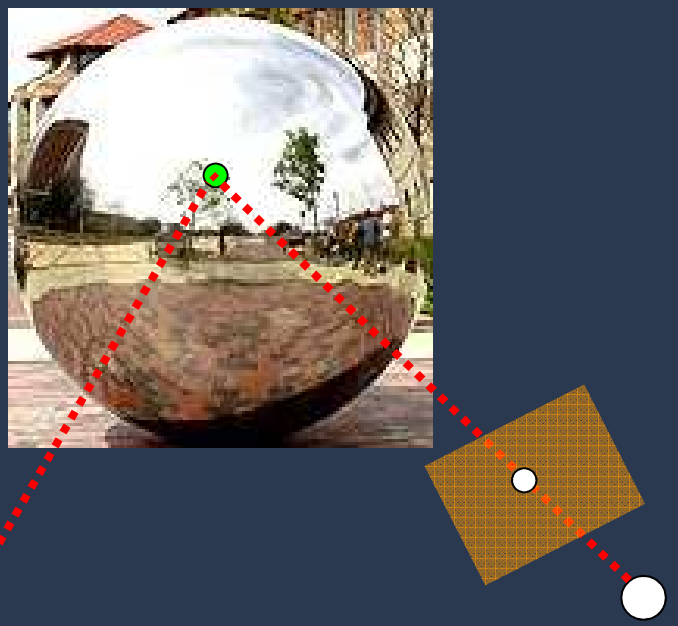
Alhazen's Problem

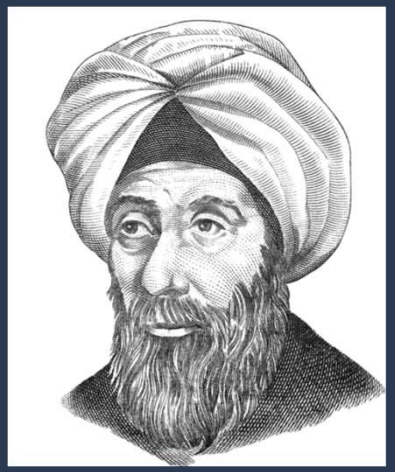


"Given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer."
- *Ptolemy*

Alhazen's Problem

- 150 A.D., Ptolemy
- 1000 A.D., Alhazen
- 1669, Barrow
- 1673, Huyghens
- 1720, L'Hopital
- 1777, Kaestner
- 1817, Hutton
- 1817, Leybourn
- 1881, Baker



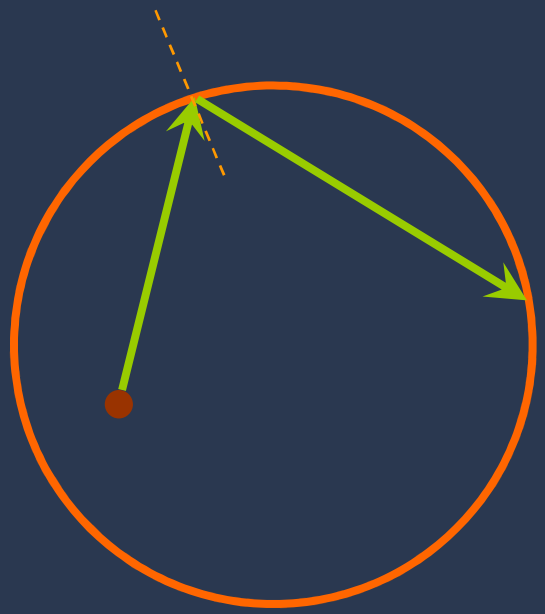


Ibn al-Haytham
(Alhazen)

Solution for Spherical Mirror

4^{th} degree equation

Also known as Circular Billiard Problem



Forward Projection for Spherical Mirror

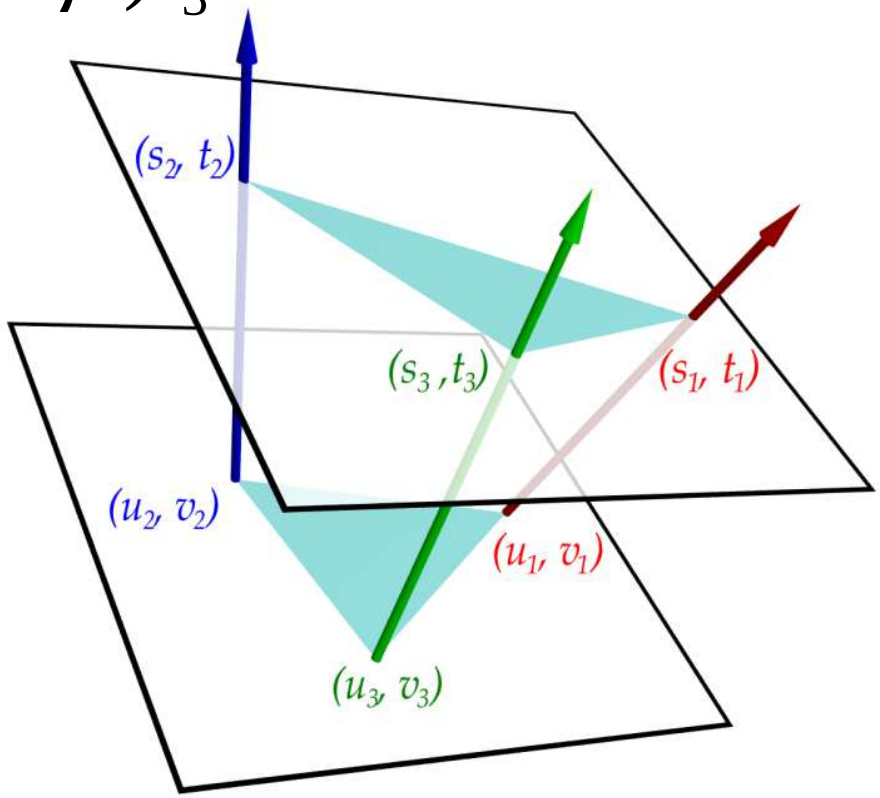
- Also known as Alhazen problem or Circular Billiard Problem
  - One of the classical problem in geometry
  - Can be traced back to Ptolemy *Optics* (AD 150)
  - Described in *Book of Optics* (~1000 A.D) by Alhazen
- References
  - Marcus Baker, “Alhazen problem”, American Journal of Mathematics, Vol. 4, No. 1, 1881, pp. 327-331
  - *Heinrich Dorrie, 100 Great Problems of Elementary Mathematics*
- Four solutions
  - Intersection of circle and hyperbola
- Forward projection for general mirrors
  - Extension of Alhazen problem

General Linear Cameras (GLC) [Yu & McMillan '04]

- **Affine combinations of 3 rays**

$$\vec{r} = \alpha\vec{r}_1 + \beta\vec{r}_2 + (1-\alpha-\beta)\vec{r}_3$$

- **Parameterized over 2 parallel planes (2PP)**
- **Each ray maps to a 4D point** $[s,t,u,v]^T$



Relative 2PP

$$\vec{r} = [\sigma, \tau, u, v]^T$$
$$\sigma = s-u,\ \tau = t-v$$

COMPUTER GRAPHICS

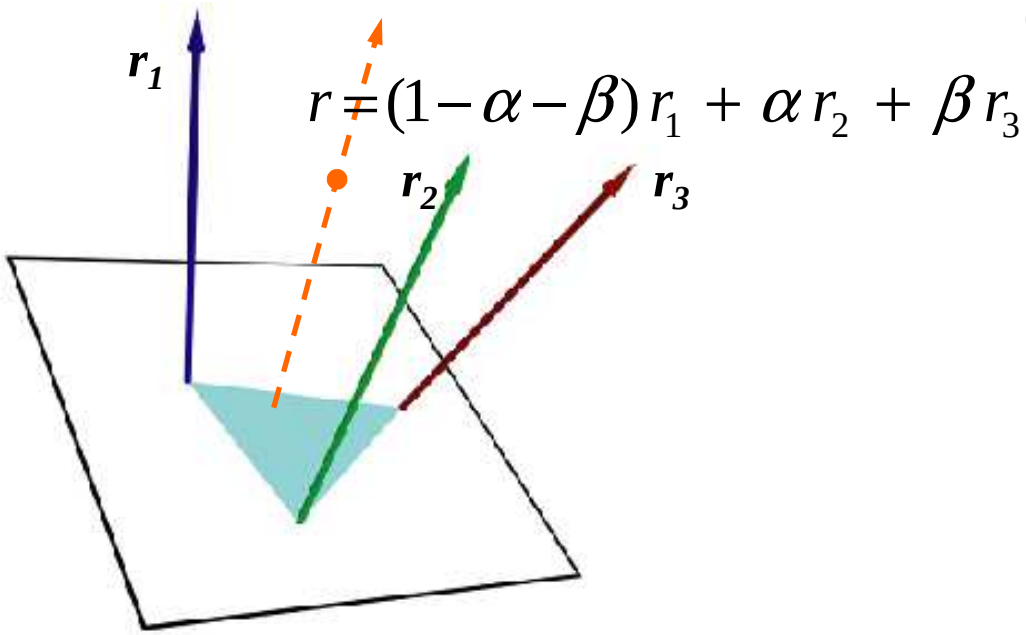
MITSUBISHI ELECTRIC

GLC Projection

- **The fundamental question:**

  How to project a 3D point to a ray in a GLC?

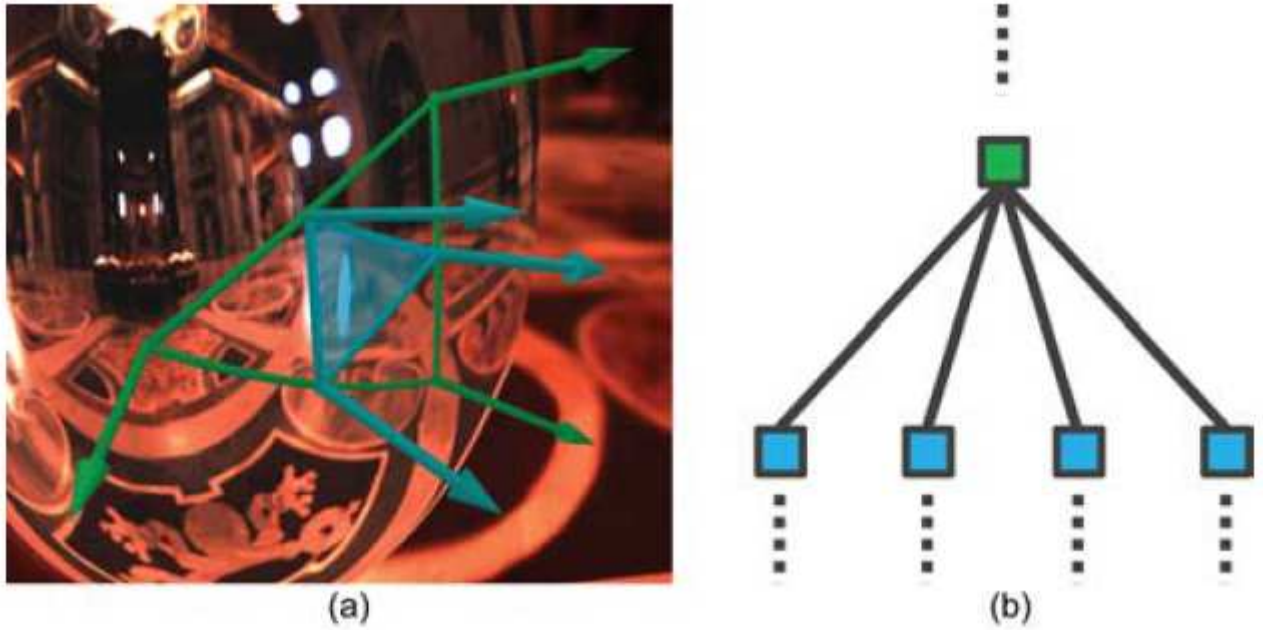
  (α, β) is the pixel coordinate



r should pass through given point P

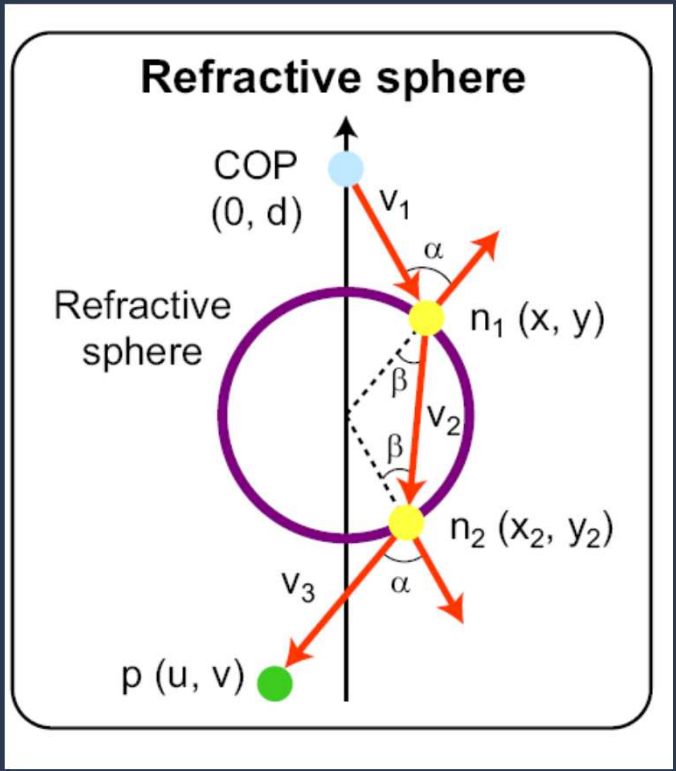
Tiling Image Space with GLCs

- **Triangulate the image plane and associate GLC with each**
- **Compute GLC projection for each GLC and check if projected pixel is within the GLC**
- Ding, Y., Yu, J., Sturm, P, Multi-perspective stereo matching and volumetric reconstruction. ICCV (2009)

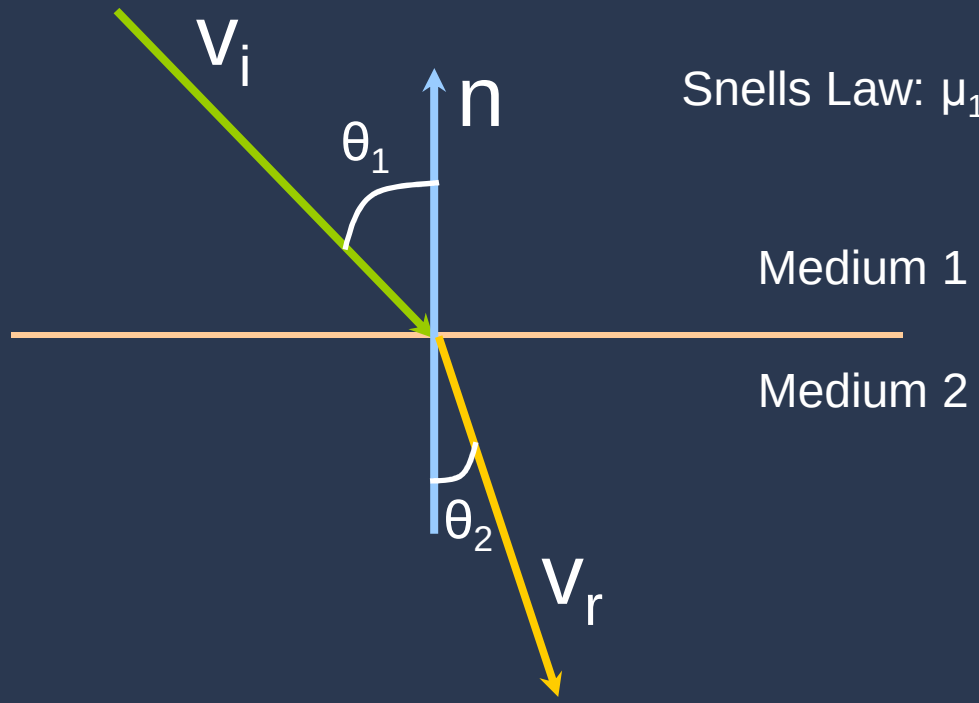


Forward Projection for Refractive Sphere

- Refractive index μ
- Key idea 1: Analysis can be done in a plane



Key idea



Snells Law: $\mu_1 \sin(\theta_1) = \mu_2 \sin(\theta_2)$

- Use vector form of the refraction equation $\mathbf{v}_r = a\mathbf{v}_i + b\mathbf{n}$

$$a = \frac{\mu_1}{\mu_2}, \quad b = \frac{-\mu_1 \mathbf{v}_i^T \mathbf{n} \pm \sqrt{\mu_1^2 (\mathbf{v}_i^T \mathbf{n})^2 - (\mu_1^2 - \mu_2^2)(\mathbf{v}_i^T \mathbf{v}_i)(\mathbf{n}^T \mathbf{n})}}{\mu_2 (\mathbf{n}^T \mathbf{n})}$$

Solution

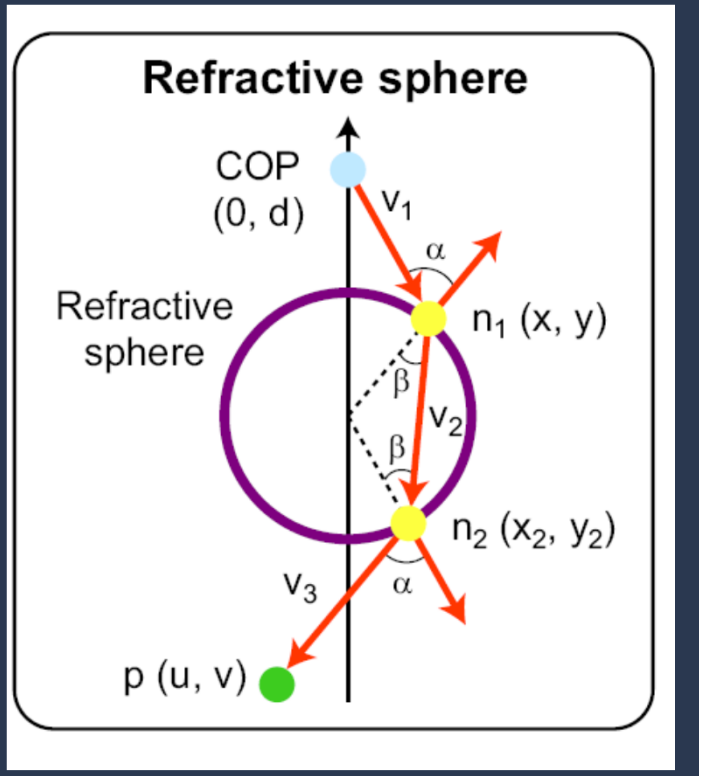
$$\mathbf{v}_1 = [x, y - d]^T$$

$$\mathbf{n}_1 = [x, y]^T$$

$$\mathbf{v}_2 = \frac{1}{\mu}\mathbf{v}_1 + \mathbf{n}_1 \frac{-\mathbf{v}_1^T\mathbf{n}_1 - \sqrt{(\mathbf{v}_1^T\mathbf{n}_1)^2 - r^2(1-\mu^2)(\mathbf{v}_1^T\mathbf{v}_1)}}{\mu r^2}.$$

$$\mathbf{v}_3 = \mu\mathbf{v}_2 + b_3\mathbf{n}_2$$

$$\mathbf{v}_3 \times (\mathbf{p} - \mathbf{n}_2) = 0$$



Solution

$$0 = K_1(x,y) + K_2(x,y)\sqrt{A} + K_3(x,y)A^{3/2}$$

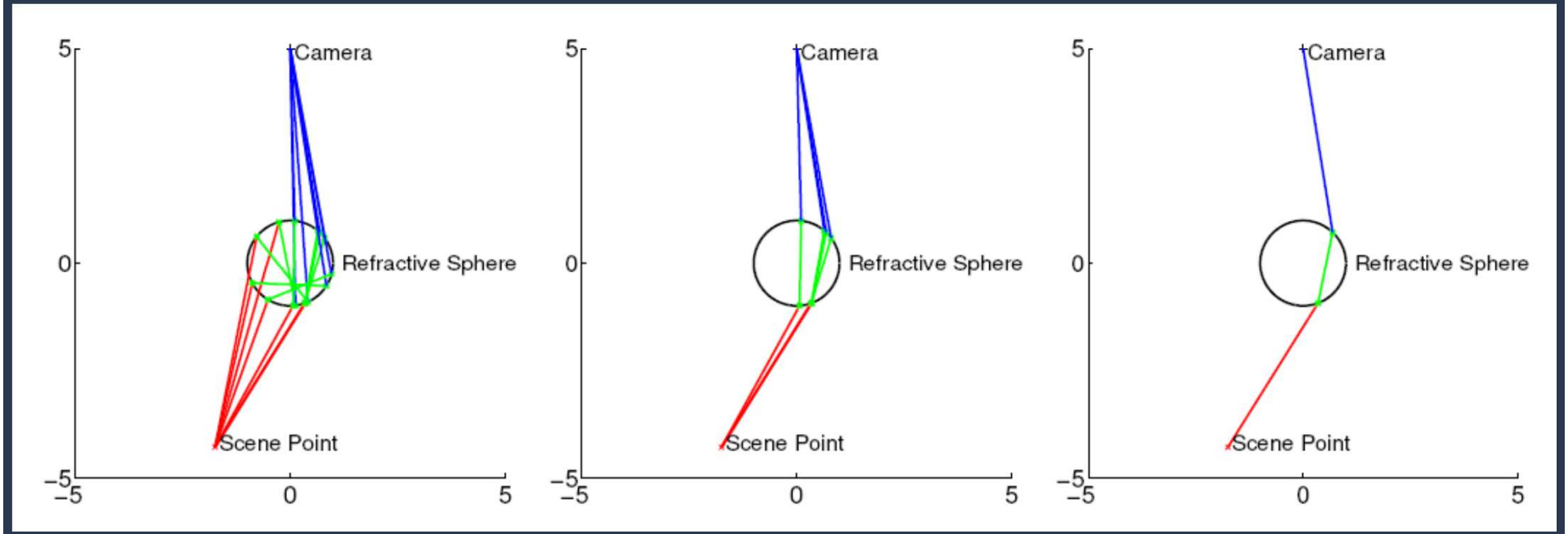
$$A = d^2\mu^2r^2 - d^2x^2 - 2d\mu^2r^2y + \mu^2r^4$$

$$x^2 = r^2 - y^2$$

10th degree equation in y

Visualizing Solutions

- Radius r=1
- Refractive Index μ = 1.5
- D = 5 (distance of camera from refractive sphere)



8 real solutions

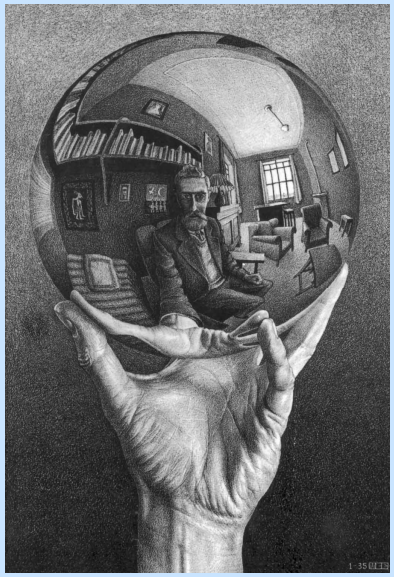
Removing invalid refraction points

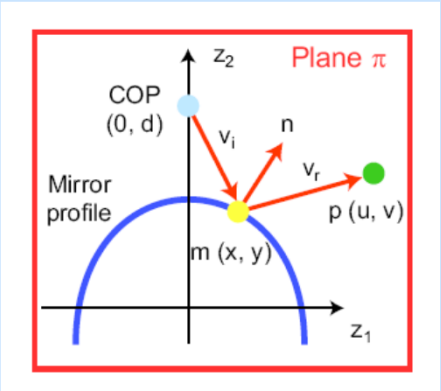
Checking Snell's Law

Analytical Forward Projection for Non-Central Cameras

Axial Catadioptric Camera

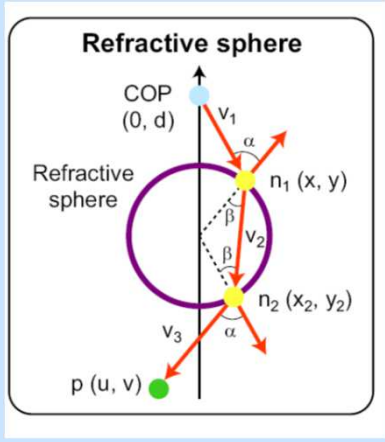
Solve 6th degree Equation

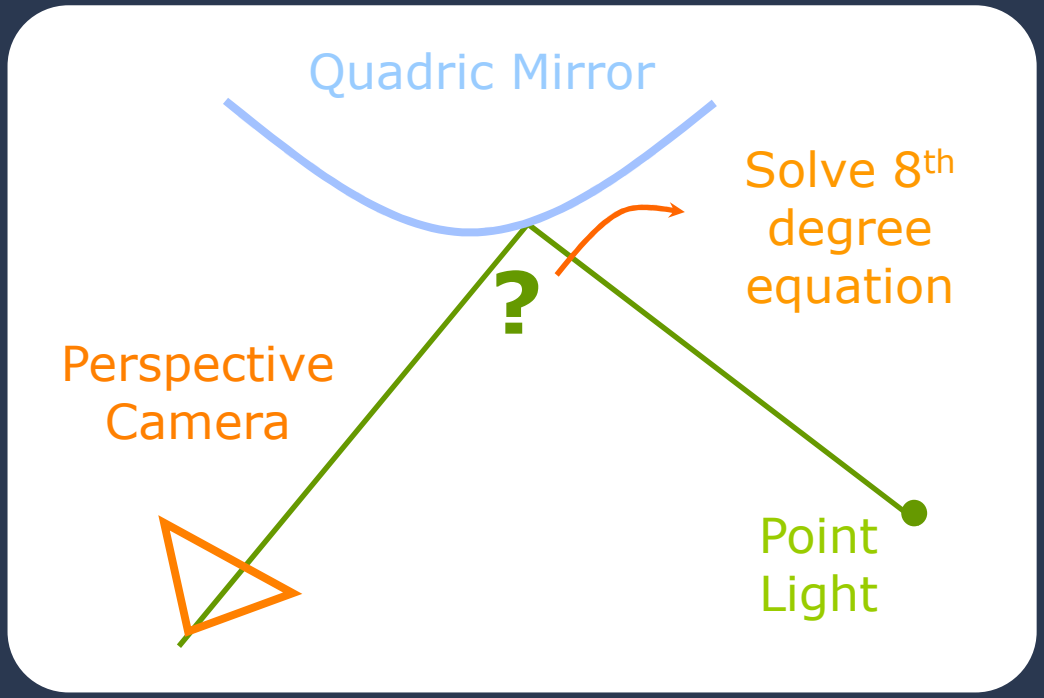




Refractive Sphere

Solve 10th degree Equation





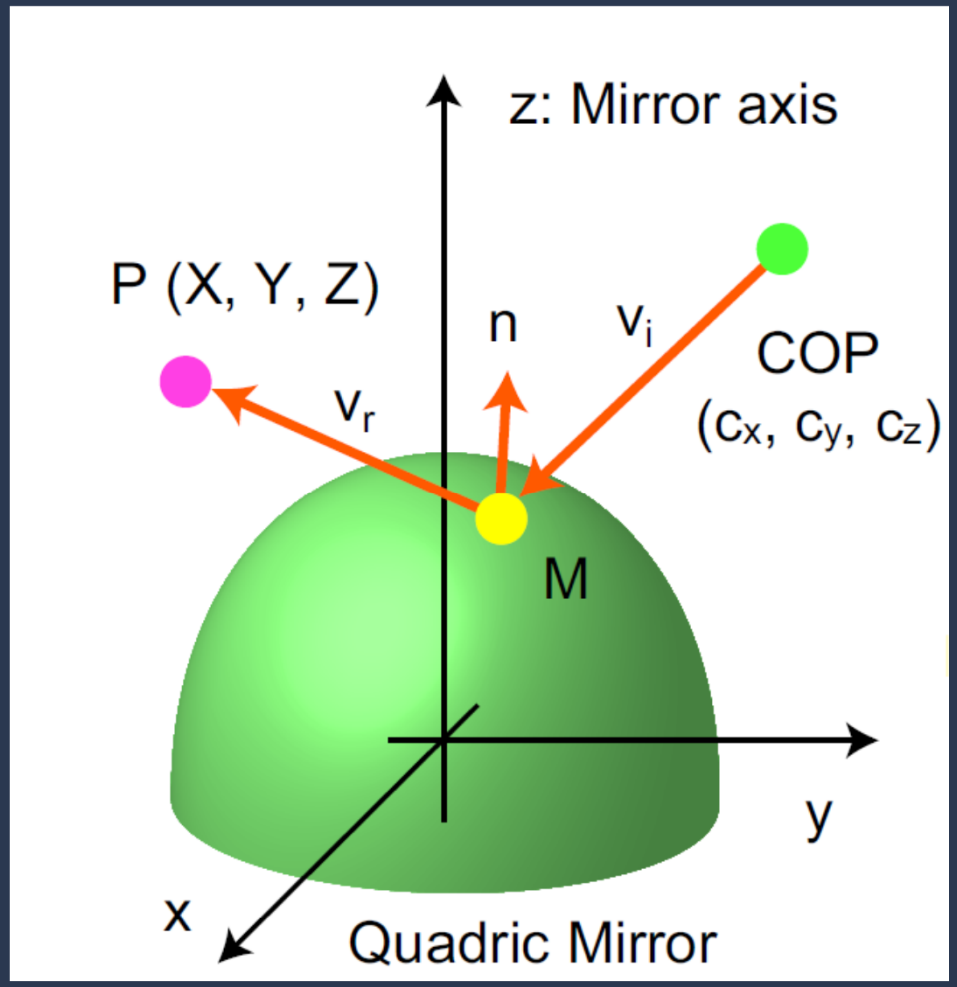
What is the point on the mirror where reflection happens?

Analytical Projection Model for Non-central Catadioptric Cameras

Beyond Alhazen's Problem

- General Quadric Mirrors
- Agrawal, Taguchi & Ramalingam, ECCV 2010
  - Solution for Axial Case
  - 6^{th} degree equation
- Bertrand Vandeportaele, 2006
  - Phd Thesis (In French)
  - Off-axis Camera + Quadric Mirrors
- Search/Optimization
  - Gonçalves, OMNIVIS 2009
  - Micusik & Pajdla, CVPR 2004

Finding the Mirror Intersection Point



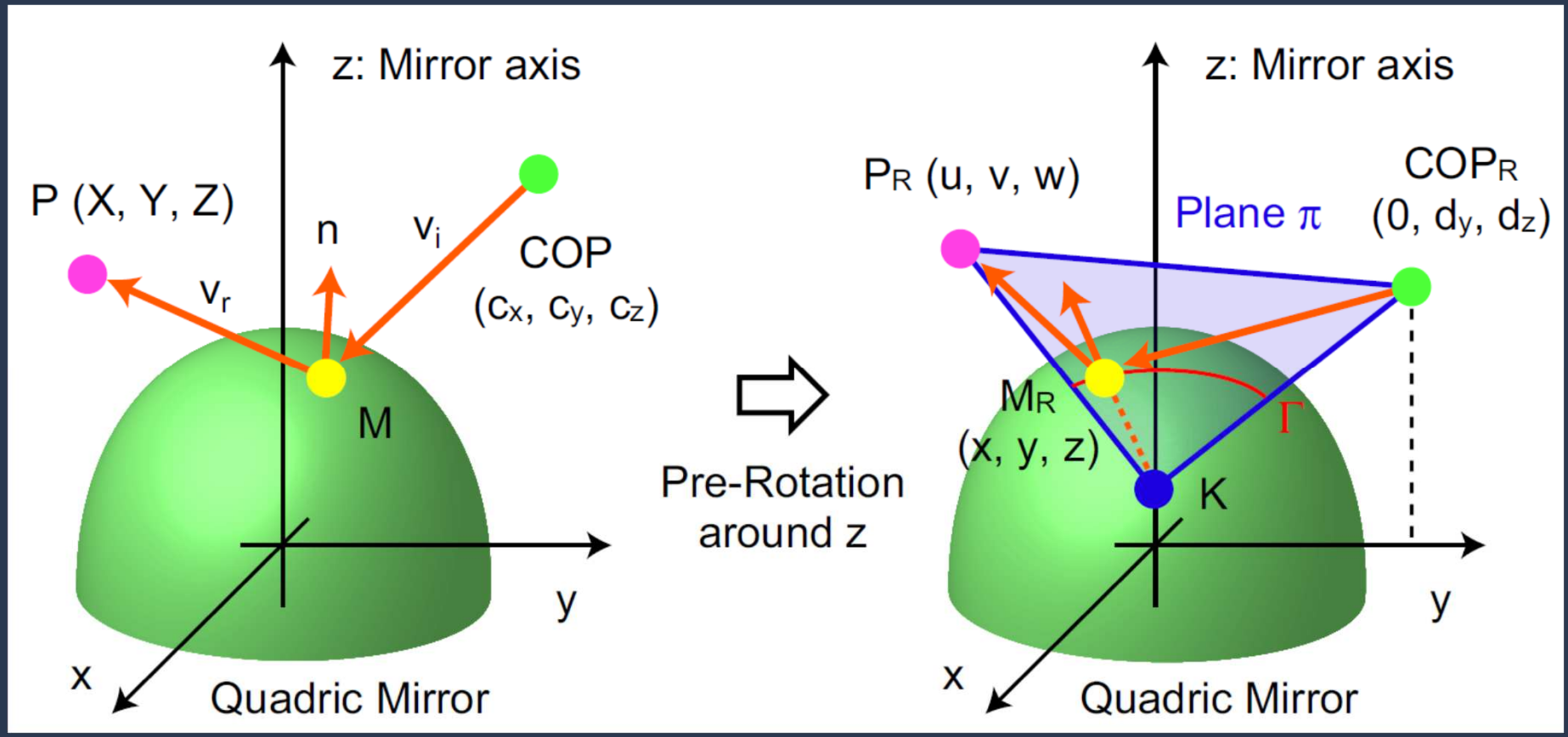
Knowns:

(A, B, C), COP, P

Unknown: M

Mirror Equation: $x^2 + y^2 + Az^2 + Bz - C = 0.$

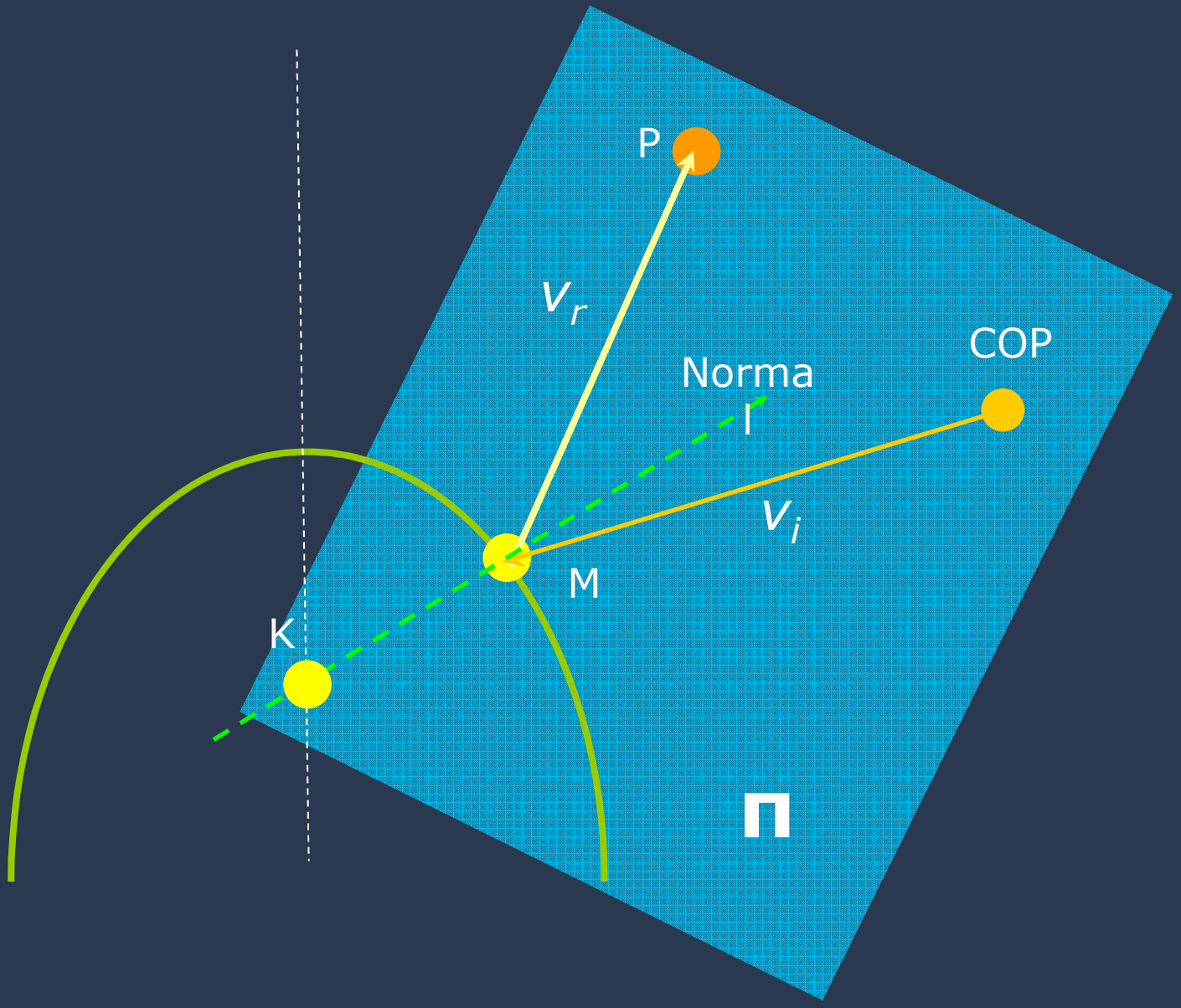
Key Idea 1: Pre-Rotation



Make the x-coordinate of COP to be zero

COP: (c_x, c_y, c_z) → (0, d_y, d_z)

Key Idea 2: Use of Reflection Plane π



Equation of Reflection Plane π

- Using P, COP and K

$$c_1(z)x + c_2(z)y + c_3(z) = 0$$

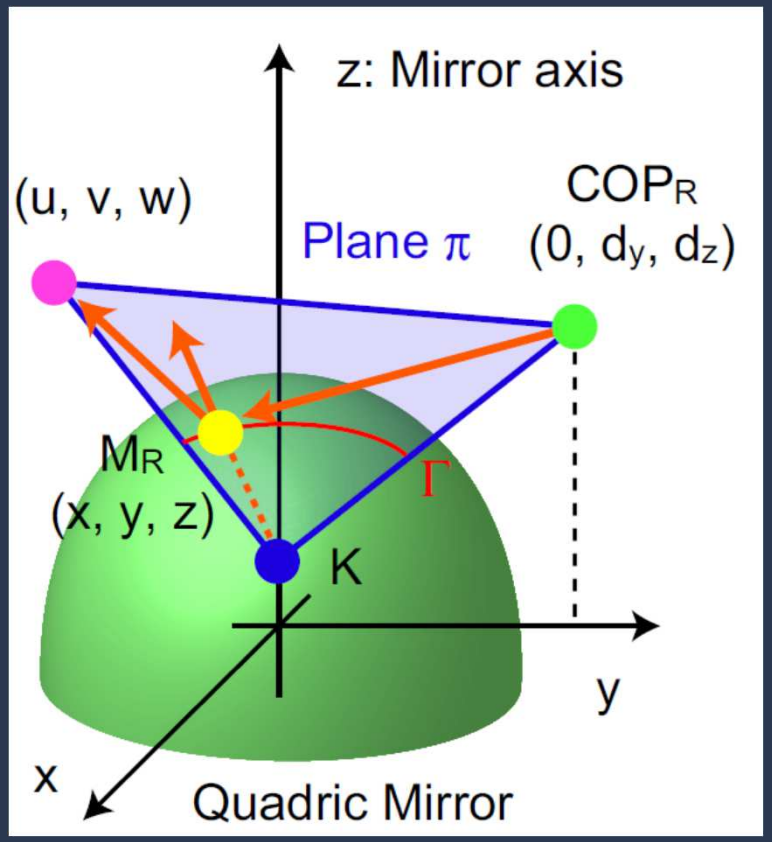
$$x = \frac{-c_2(z)y - c_3(z)}{c_1(z)}$$

Substitute in Mirror Equation

$$\mathbf{IE}_1 : k_{41}(z)y^2 + k_{42}(z)y + k_{43}(z) = 0$$

$$\mathbf{IE}_1 : k_{41}(z)y^2 + k_{42}(z)y + k_{43}(z) = 0$$

- Intersection of Reflection plane π with the Mirror
- Curve Γ on the mirror where reflection can happen



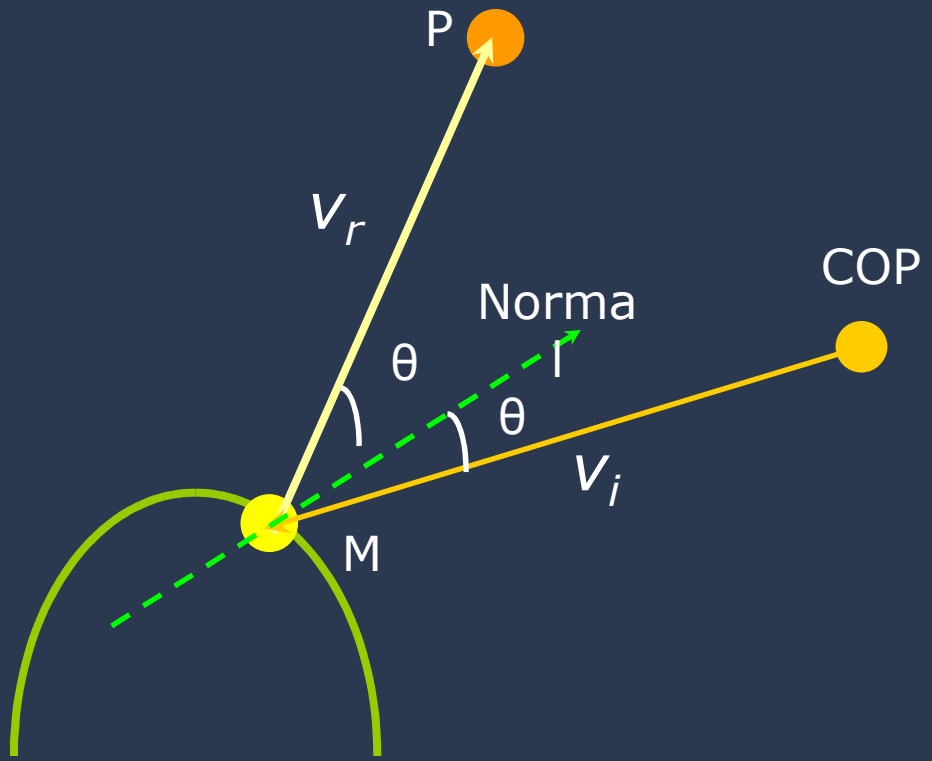
Law of Reflection

$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{v}_i^T\mathbf{n})/(\mathbf{n}^T\mathbf{n})$$

V_r should be same as (P-M)

cross(vr, P-M) = 0

$$\mathbf{IE}_2 : k_{31}(z)y^2 + k_{32}(z)y + k_{33}(z) = 0$$



Final Equation

- Two quadratic equations in y

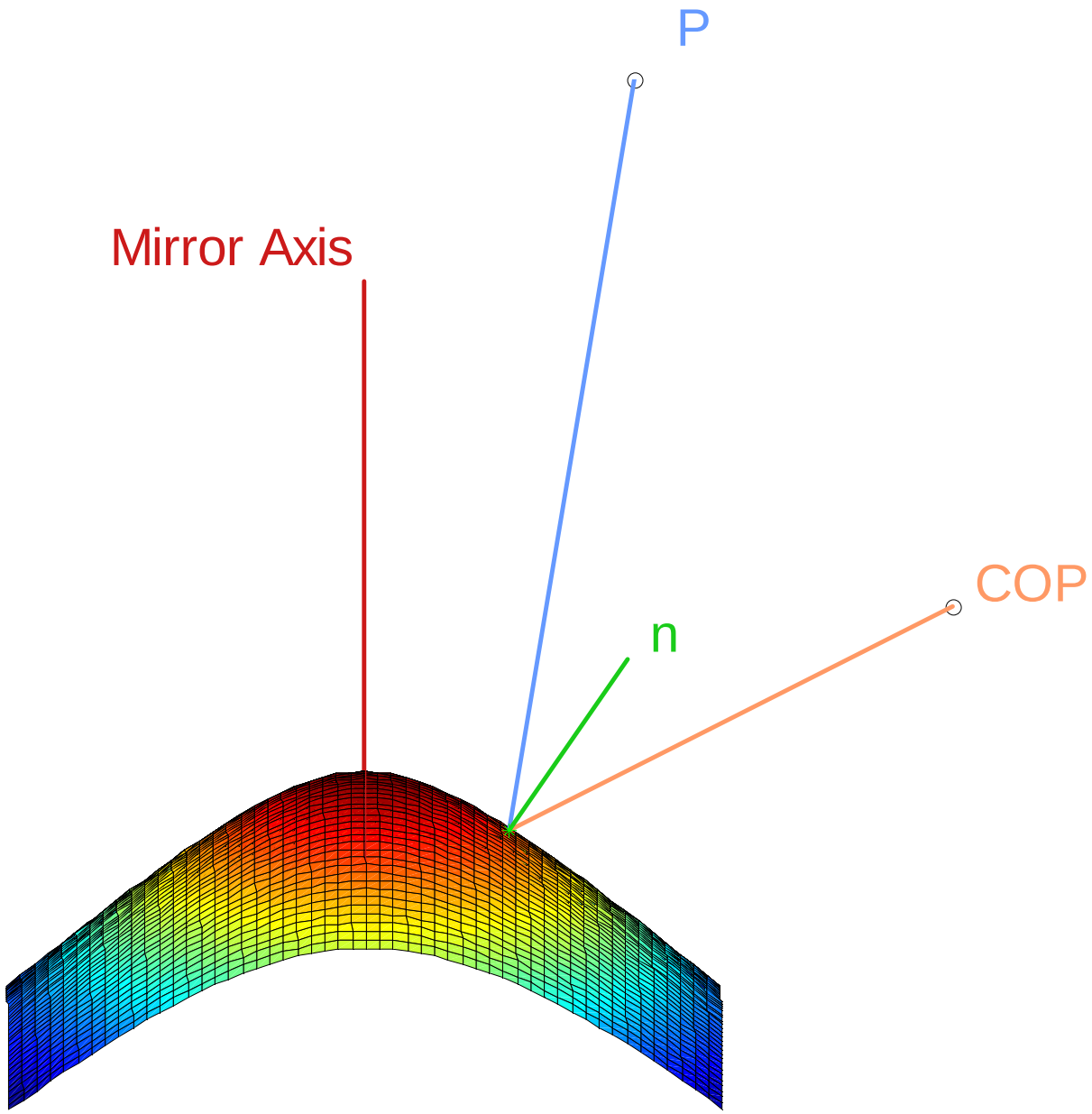
$$\mathbf{IE}_1 : k_{41}(z)y^2 + k_{42}(z)y + k_{43}(z) = 0$$

$$\mathbf{IE}_2 : k_{31}(z)y^2 + k_{32}(z)y + k_{33}(z) = 0$$

$$\begin{aligned} & k_{41}(z)\left(k_{43}(z)k_{32}^2(z) - k_{42}(z)k_{32}(z)k_{33}(z) + k_{41}(z)k_{33}^2(z)\right) \\ & - k_{31}(z)(-k_{33}(z)k_{42}^2(z) + k_{43}(z)k_{32}(z)k_{42}(z) \\ & + 2k_{41}(z)k_{43}(z)k_{33}(z)) + k_{43}^2(z)k_{31}^2(z) = 0. \end{aligned}$$

- 8^{th} degree equation in z

Mirror Shape	Parameters	Camera Placement		
		Off-Axis	Axial	
			NSVP	SVP
General	A, B, C	8	6	-
Spherical	$A = 1, B = 0, C > 0$		$4^{\ddagger}$	-
Elliptical	$A > 0, B = 0, C > 0$	8	6	2
Hyperbolic	$A < 0, B = 0, C < 0$	8	6	2
Parabolic	$A = 0, C = 0$	7	5	$2^{\dagger}$
Conical	$A < 0, B = 0, C = 0$	4	2	-
Cylindrical	$A = 0, B = 0, C > 0$	4	2	-



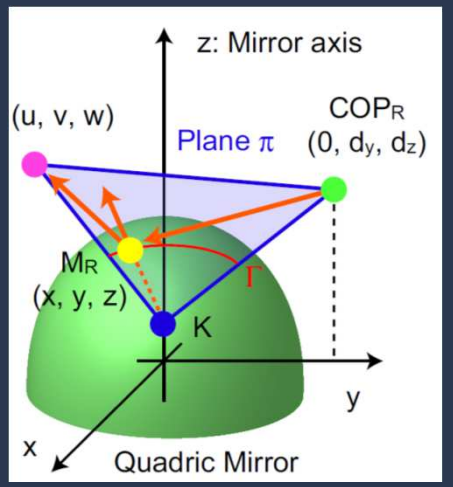
Hyperbolic Mirror

Other Approaches

- Intersection of Reflection plane π with the Mirror
- Curve Γ on the mirror where reflection can happen
- 1D search on this curve or

Optimization on single variable

- Goncalvez, N., Nogueira, A.C.: Projection through quadric mirrors made faster. OMNIVIS (2009)



Advantage: Fast Projection of 3D Points

- Off-axis camera looking at quadric mirror
- Project randomly generated 100,000 points
- Matlab on standard PC
- 40X speed up compared to optimization
  - Micusik and Pajdla, CVPR 2004
- 100X speed up for spherical mirrors

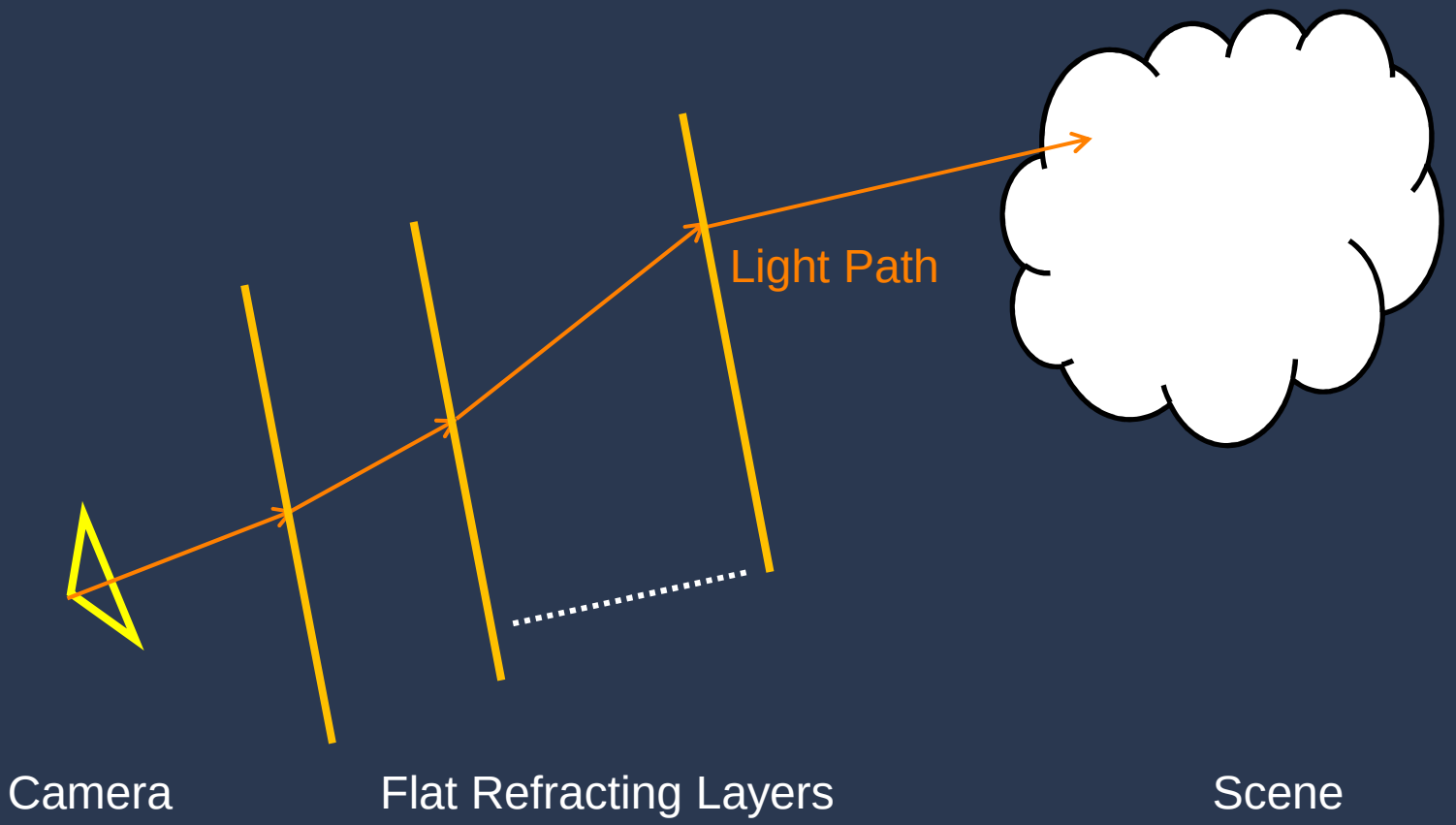
Summary

- Forward projection for quadric mirrors and refractive sphere
  - Optimization, Search
  - GLC -- generic solution, but slow
  - Analytical solutions -- fast, but specific to problem

Imaging through refractions

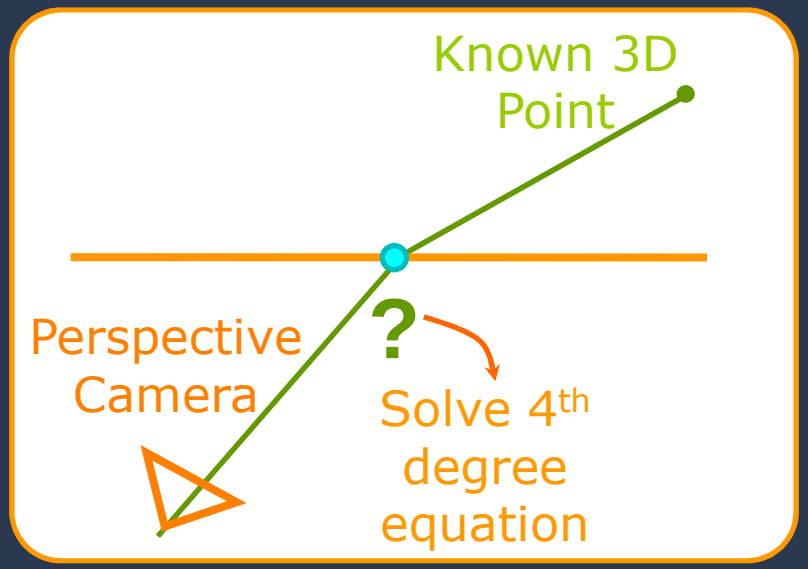
- Not the same as pinhole imaging
- Pinhole model (central approximation) is not valid

Multi-Layer Flat Refractive Systems



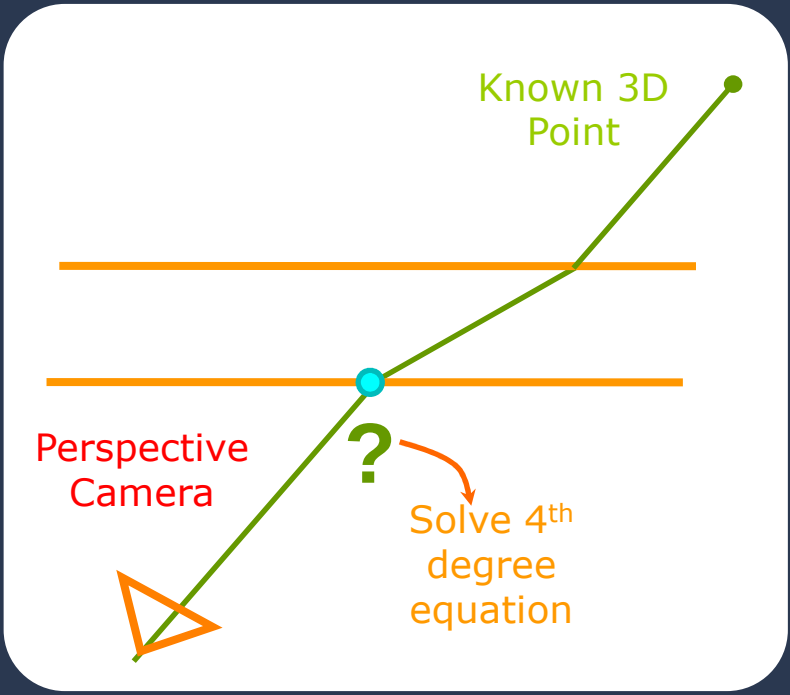
Analytical Forward Projection

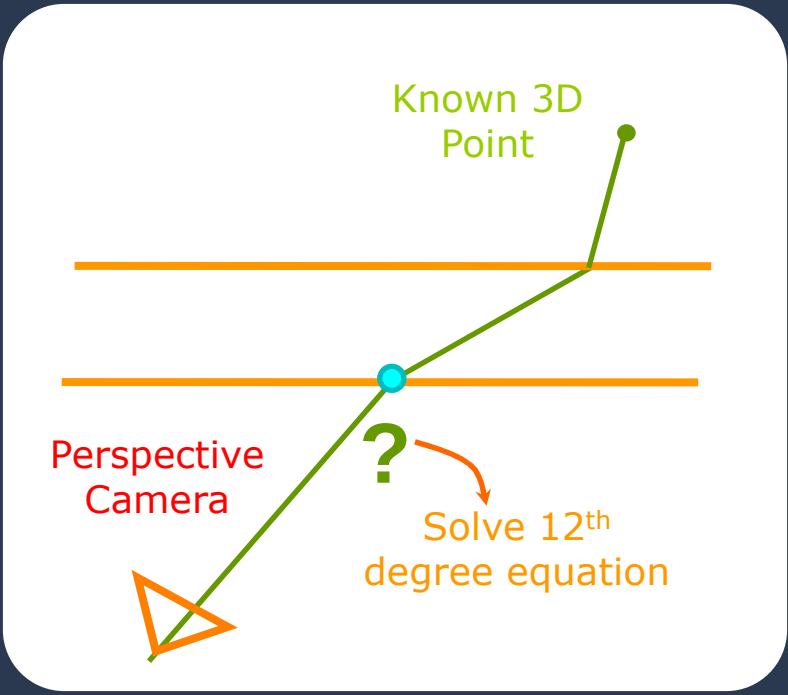
- Single Layer
  - 4th degree equation
  - Glaeser and H.-P.Schrocker. Reflections on refractions, *J. Geometry and Graphics*, 4(1):1–18, 2000



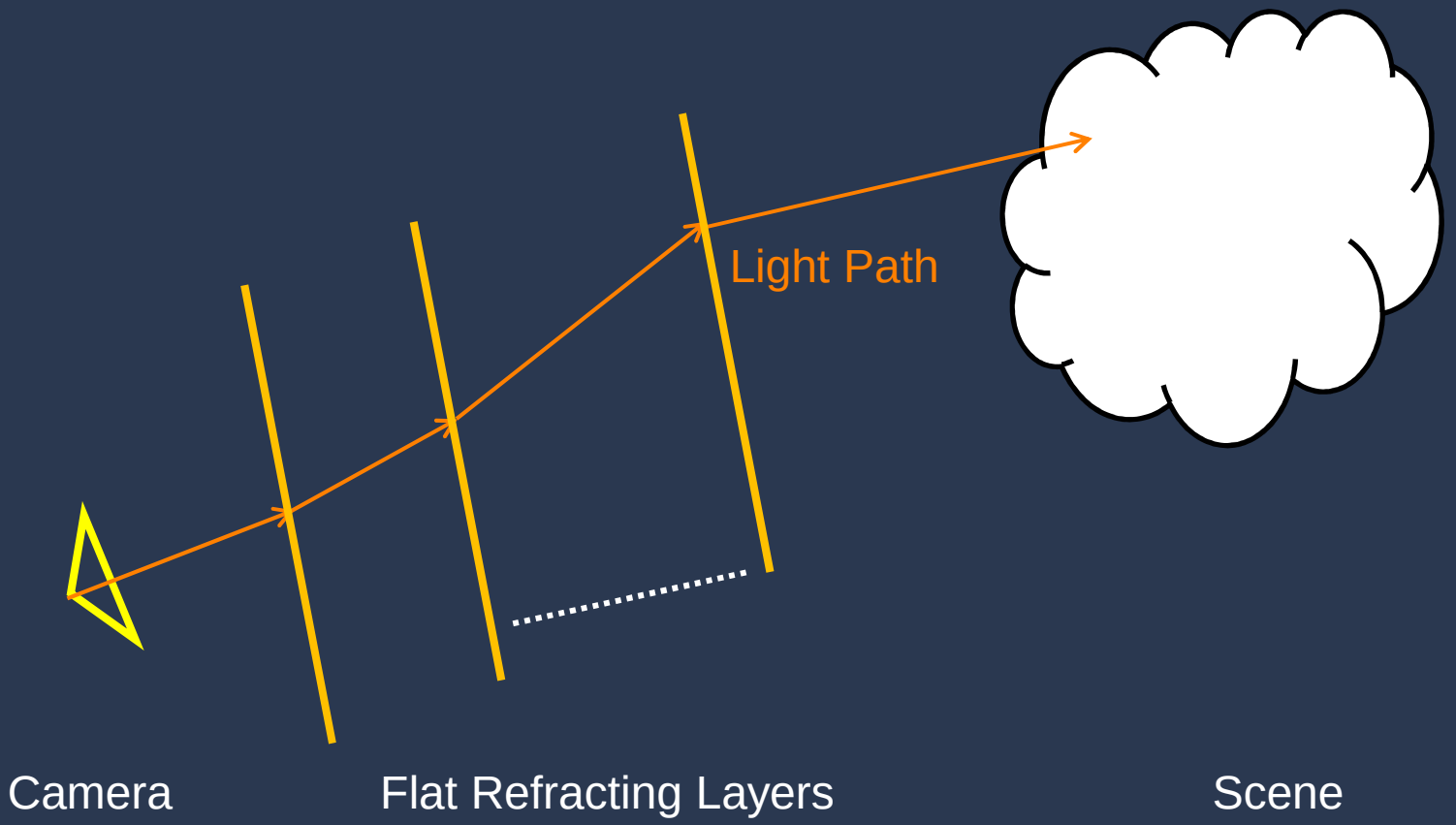
Analytical Forward Projection

- Two Layers
  - Air – Medium – Air 4^{th} degree equation
  - General Case 12^{th} degree equation

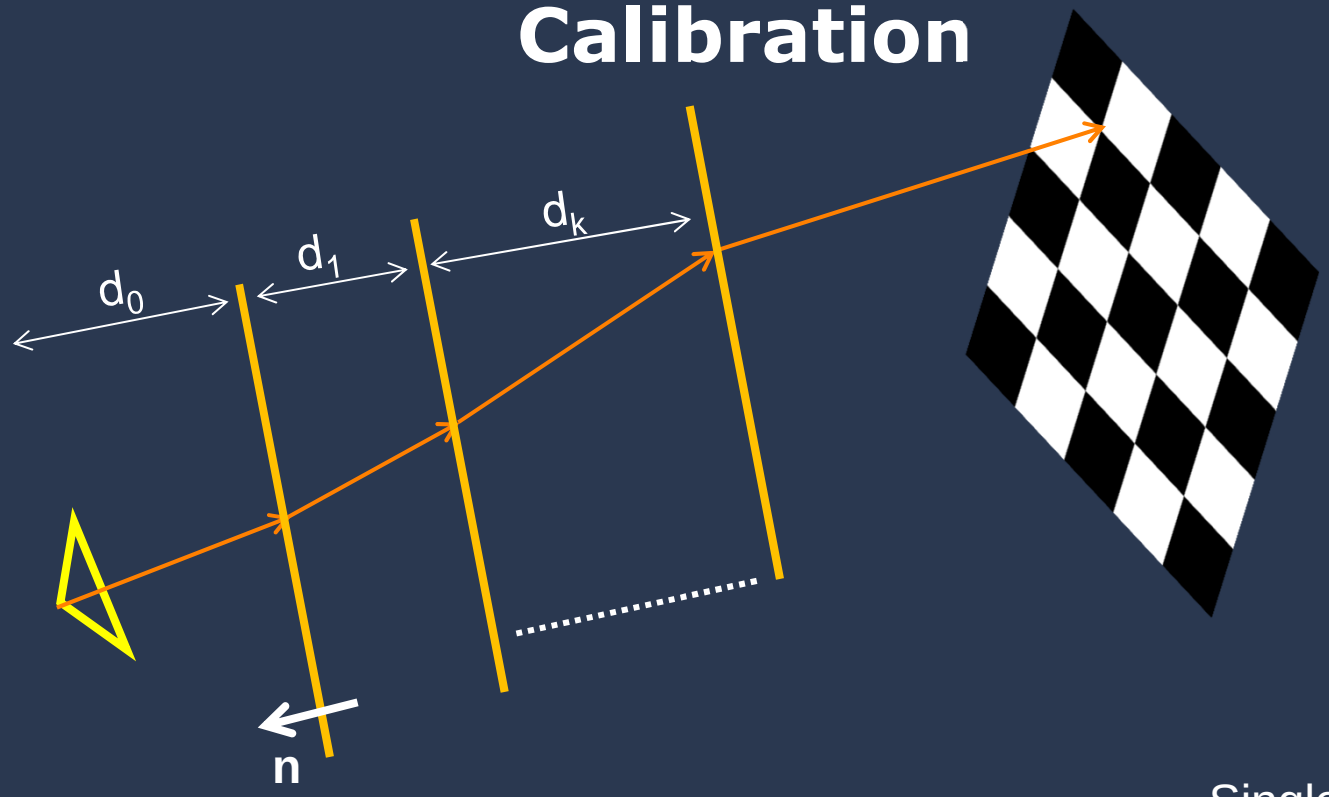




Multi-Layer Flat Refractive Systems



Calibration



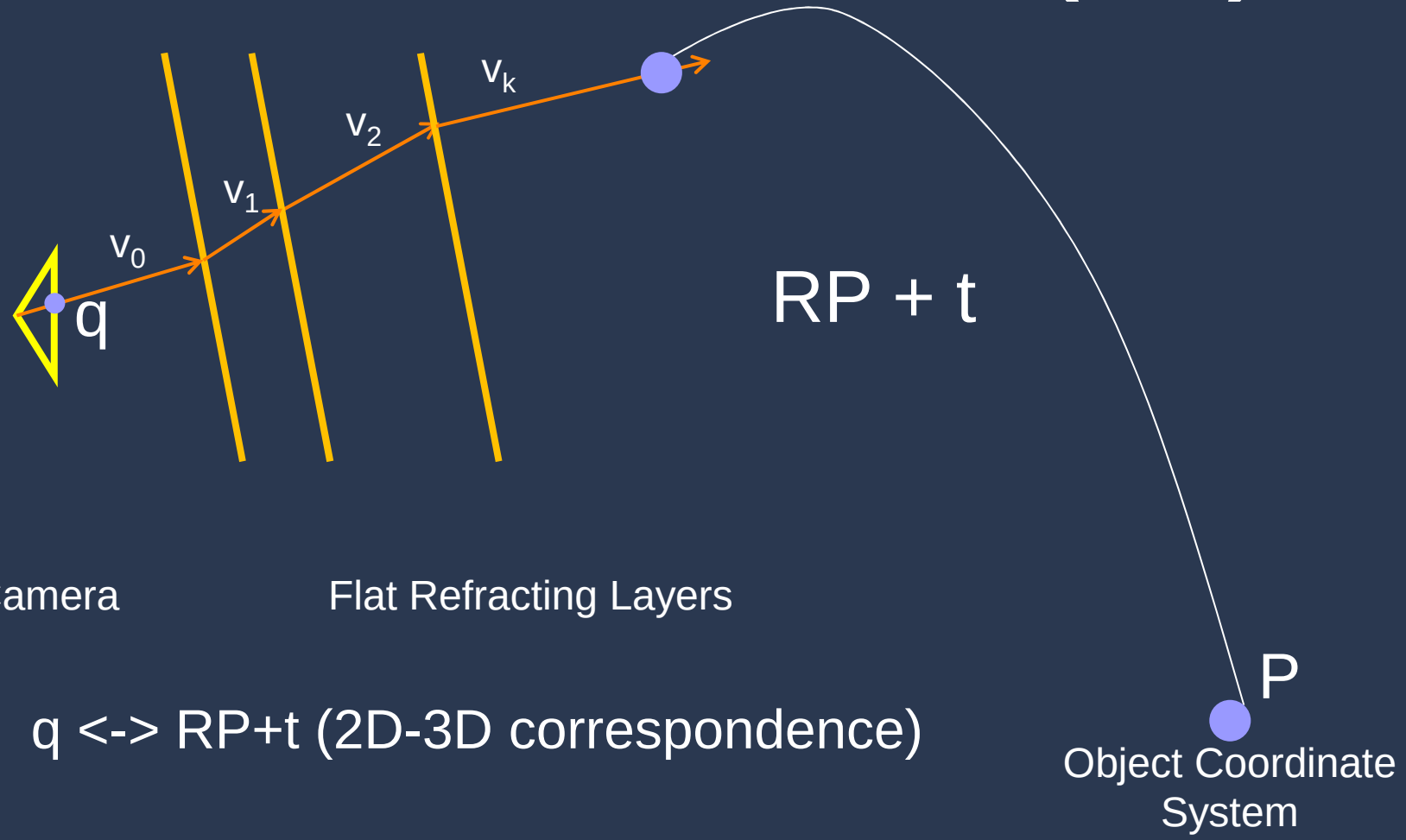
Camera

Flat Refracting Layers

Single Planar Checkerboard

-Unknown Orientation of Layers **n**
-Unknown Layer Thickness d_0, d_1, … d_{k-1}
-Unknown Refractive Indices μ_0, μ_1, …, μ_{k-1}
-Unknown Pose of the Checkerboard (R,t)

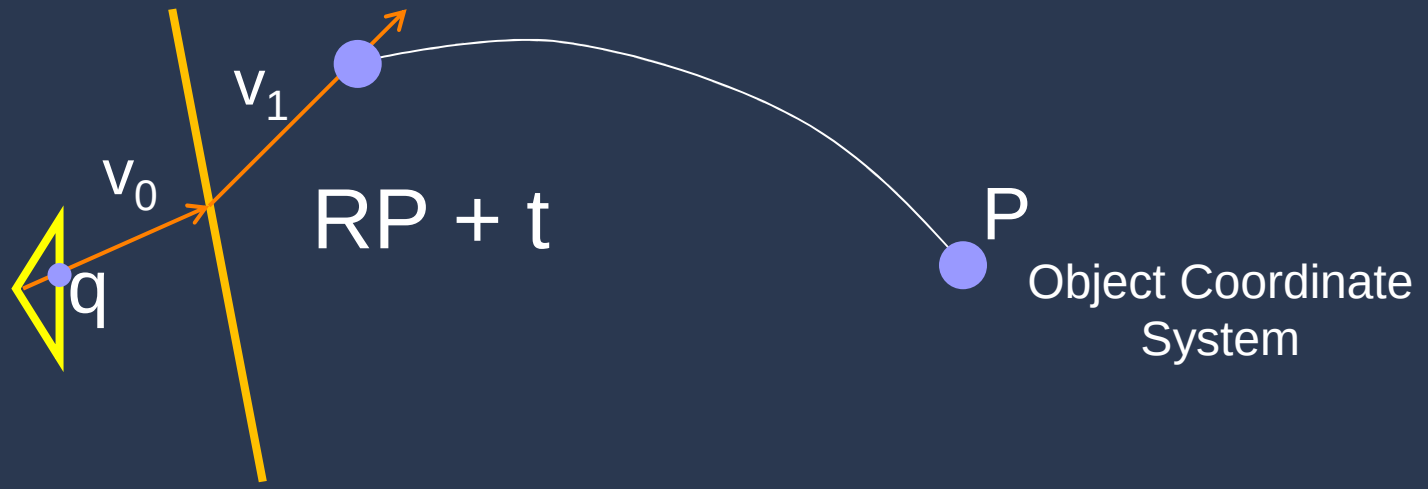
Flat Refraction Constraint (FRC)



q <-> RP+t (2D-3D correspondence)

Transformed 3D point (RP+t) should lie on the outgoing ray v_k

FRC for Single Layer

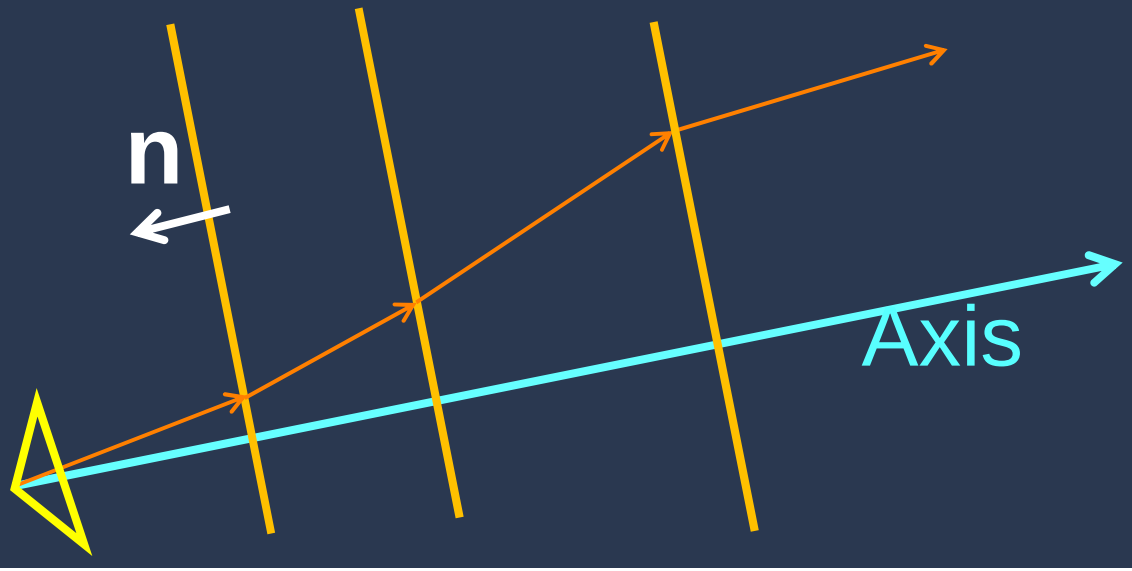


- Non-Linear Equation in 10 unknowns
- Difficult to solve
- Complexity increases with each additional layer

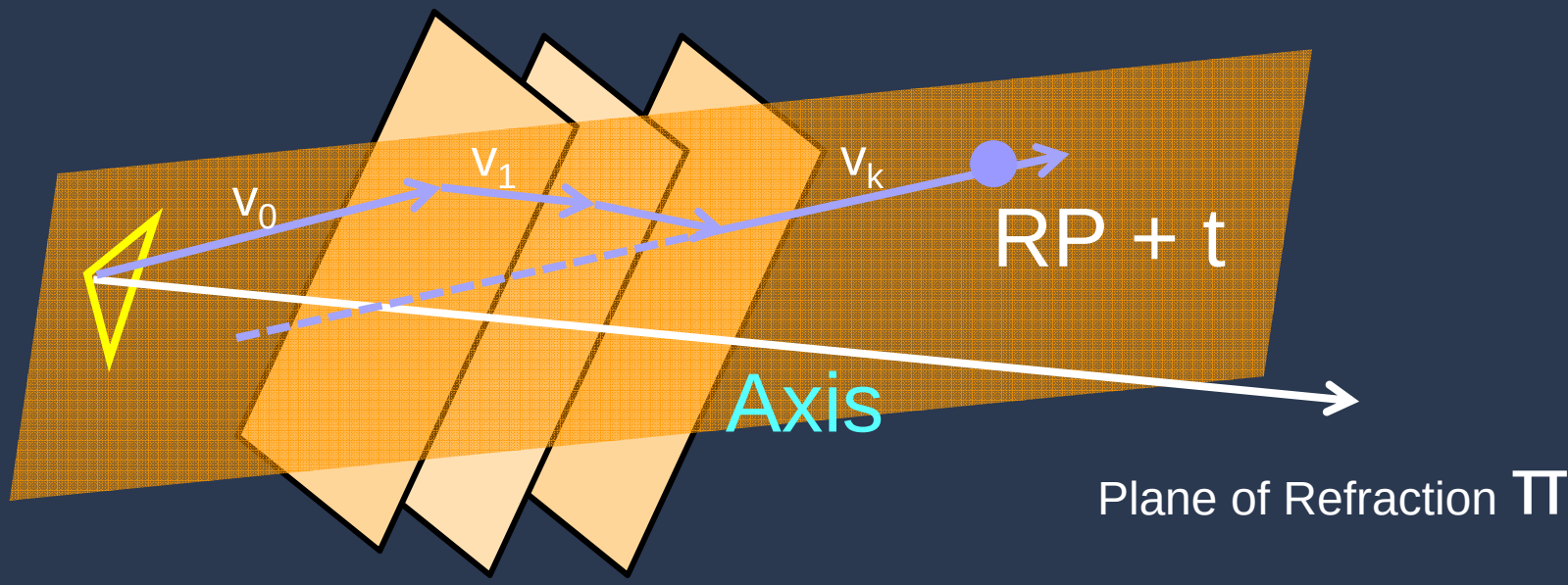
Modeling Flat Refractions

- Pinhole Model is not good
  - Non-single view point camera
  - Well-known in photogrammetry (Kotowski 1988)
  - Treibitz et al. CVPR 2008

- Flat Refraction corresponds to Axial (non-central) camera
  - All outgoing rays pass through an axis
  - Axis: Camera ray parallel to layer orientation **n**



Flat Refraction == Axial Camera



Transformed 3D point (RP + t) should also lie on the plane of refraction

$$(R\mathbf{P} + \mathbf{t})^T (\mathbf{A} \times \mathbf{v}_0) = 0$$

Axial Cameras: Coplanarity Constraint

- Transformed 3D point (RP+t) should lie on plane of refraction
  - Weaker constraint than FRC
- Axis A, Camera ray v_0

$$(R\mathbf{P} + \mathbf{t})^T (\mathbf{A} \times \mathbf{v}_0) = 0$$

- Independent of number of layers, layer distances and their refractive indices
- Allows estimating axis and pose independently of other calibration parameters

Coplanarity Constraint

$$(R\mathbf{P} + \mathbf{t})^T (\mathbf{A} \times \mathbf{v}_0) = 0$$

$$E = [\mathbf{A}]_\times R \text{ and } \mathbf{s} = \mathbf{A} \times \mathbf{t}.$$

$$\mathbf{v}_0^T E\mathbf{P} + \mathbf{v}_0^T \mathbf{s} = 0$$

- Translation along axis vanishes in **s**
- 5 out of 6 pose parameters can be computed

11 Point Linear Algorithm

$$v_0^T E_{3x3} P + v_0^T s_{3x1} = 0$$

Using 11 2D-3D correspondences, we get 11 by 12 matrix **B**

$$\underbrace{\begin{bmatrix} (\mathbf{P}(1)^T \otimes \mathbf{v}_0(1)^T) & \mathbf{v}_0(1)^T \\ \vdots & \vdots \\ (\mathbf{P}(11)^T \otimes \mathbf{v}_0(11)^T) & \mathbf{v}_0(11)^T \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} E(:) \\ \mathbf{s} \end{bmatrix} = 0$$

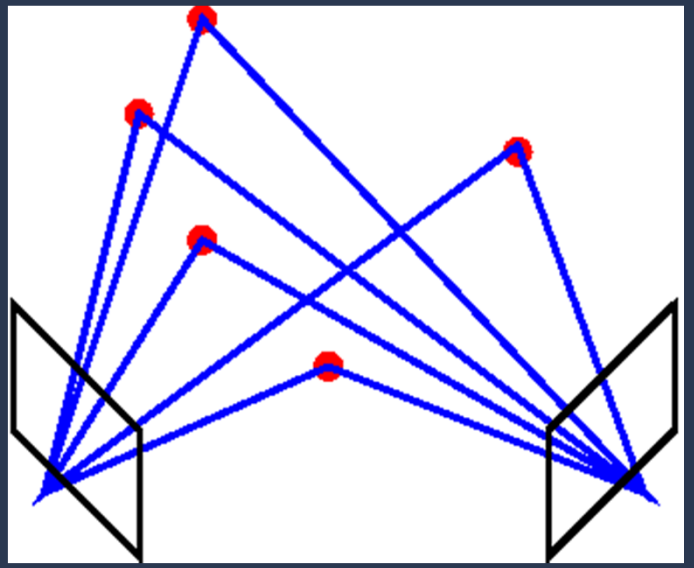
SVD based solution

Similarity with 5-point Relative Pose Problem

- $E = [A]_x R$, where A is the axis and R is unknown rotation

- For relative pose between two cameras
  - Essential matrix $E = [t]_x R$, where t is the translation
  - 5-point algorithm [Nister 2004]

- We can map our problem to the 5-point Relative Pose problem



8-Point Axis Estimation Algorithm

$$v_0^T E P + v_0^T s = 0$$

- Using 8 correspondences, we get 8 by 12 matrix **B**
- Solution lies in 4 dimensional sub-space

$$\begin{bmatrix} E(:) \\ \mathbf{s} \end{bmatrix} = \lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2 + \lambda_3 \mathbf{V}_3 + \lambda_4 \mathbf{V}_4$$

$$E(:) = \lambda_1 \mathbf{V}_1(1:9) + \lambda_2 \mathbf{V}_2(1:9) + \lambda_3 \mathbf{V}_3(1:9) + \mathbf{V}_4(1:9)$$

Feed subspace vectors to Nister's Solver and obtain λ_i

8-Point Axis Estimation Algorithm

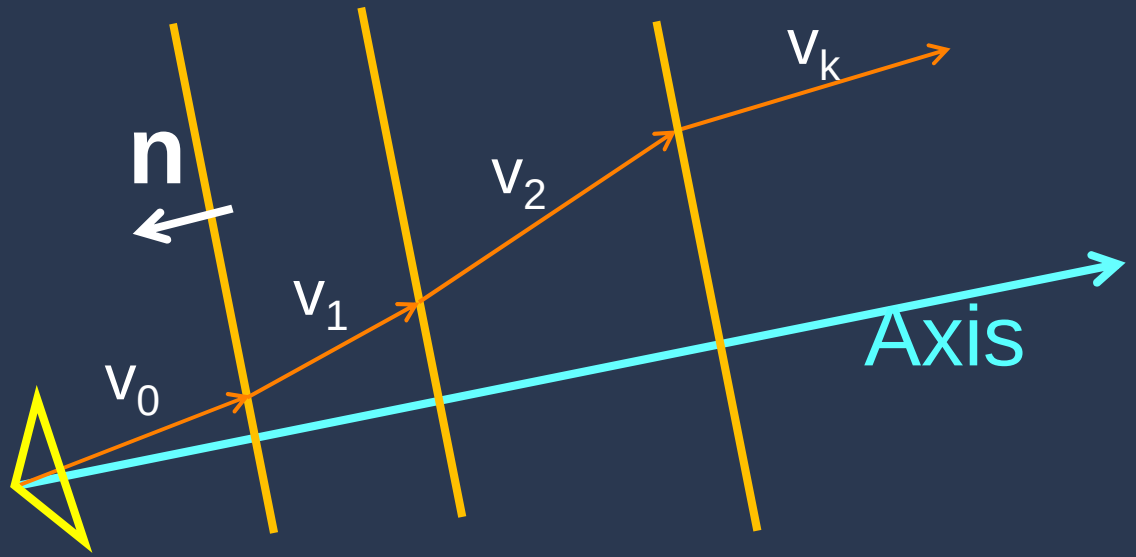
- Compute Axis from E as left null-singular vector
  - $A^T E = 0$
- Compute Rotation matrices from E
  - Hartley and Zisserman, Multiview Geometry
  - Twisted pair ambiguity
  - Similar to Relative Pose problem

Obtaining Remaining Calibration Parameters

- Coplanarity Constraint
  - Obtain axis A, rotation R, and s = A×t

- Remaining calibration parameters
  - Translation along axis t_A
  - Layer Thickness d_i, i = 1 to k
  - Layer Refractive Indices μ_i, i = 1 to k

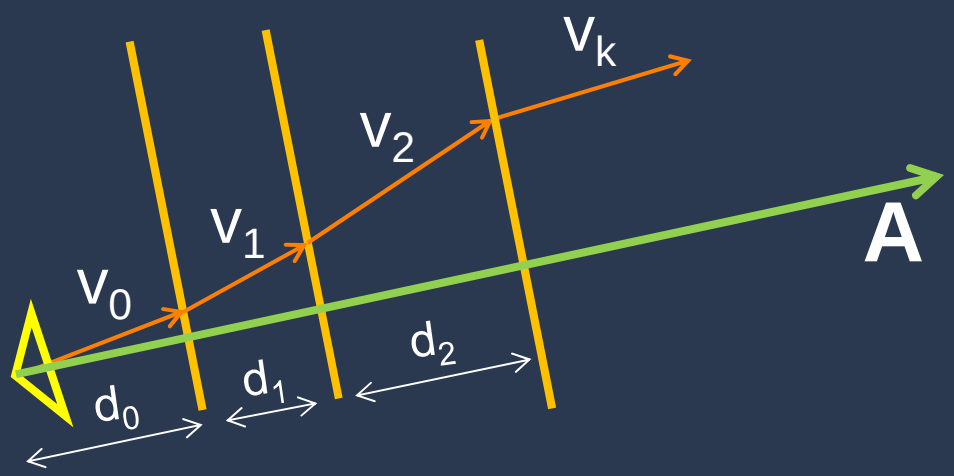
Known Refractive Indices



- Ray directions of $v_1, \ldots v_k$ can be computed using Snell's Law
- Layer Thicknesses d_i and t_A can be computed linearly

Linear System

$$\begin{pmatrix} v_k \times v_0 & v_k \times v_1 & \cdots & v_k \times v_{k-1} & v_k \times A \end{pmatrix} \begin{pmatrix} d_0 \\ \vdots \\ d_{k-1} \\ t_A \end{pmatrix} = \mathbf{u}$$



Calibration: Summary

- For Axial cameras, co-planarity constraint allows computing the axis and pose of the checkerboard
  - Remaining calibration parameters can be computed depending on the specific imaging system